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On the leaf spaces of singular holomorphic foliations and multiplicities on leaves

THESIS

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A mes grand-parents

“La musique est peut-être l’exemple unique de ce qu’aurait pu être - s’il n’y avait pas eu l’invention du langage, la formation des mots, l’analyse des idées - la communication des âmes.”
Marcel Proust

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Denis Morel

Résumé

Nous nous intéressons à l'espace des feuilles $X/\mathbb{F}$ d'un feuilletage holomorphe $\mathbb{F}$, qui peut être régulier ou avec singularités, sur une variété complexe X. Nous allons apporter des éléments de solution aux deux problèmes suivants à l'aide de la théorie des cycles analytiques, théorie introduite par Barlet:

Premier problème *Trouver des conditions suffisantes qui impliquent que, pour un feuilletage avec des feuilles partout, l'espace des feuilles est un espace complexe.*

Deuxième problème *Que peut-on faire si le feuilletage n'a pas des feuilles partout, ou bien si il a des feuilles partout, mais l'espace des feuilles n'est pas un espace complexe ?*

Nous nous intéressons d'abord au premier problème, ceci dans le cas de feuilletages réguliers.

Nous définissons la notion de multiplicité topologique $\mu_t(L)$ pour certaines feuilles L du feuilletage. Nous définissons ensuite une application $\zeta_{\mathbb{F}}\colon G(\mathbb{F}) \longrightarrow Z_d(X)$, où $G(\mathbb{F})$ est l'ensemble des feuilles dont la multiplicité topologique est bien définie et $Z_d(X)$ est l'espace topologique des cycles analytiques de dimension d (d est la dimension du feuilletage) muni de la topologie de Barlet. Cette application associe à chaque point $x \in G(\mathbb{F})$ le cycle $\mu_t(L_x)L_x$. Nous démontrons dans le théorème 6.1.1 les équivalences suivantes:

$X/\mathbb{F}$ est un espace complexe $\Longleftrightarrow$ $G(\mathbb{F}) = X$ et $\zeta_{\mathbb{F}}$ est continue

$\Longleftrightarrow$ Il existe une application continue et $\mathbb{F}$-saturée $\varphi\colon X \longrightarrow Z_d(X)$ telle que pour tout x le support de $\varphi(x)$ est L_x .

$\Longleftrightarrow$ L'application canonique $X \to \widetilde{Z}_d(X)$ qui associe à chaque x la feuille qui passe par x est continue ($\widetilde{Z}_d(X)$ est l'ensemble des sous-ensembles analytiques de dimension d de X muni de la topologie appelée topologie de Barlet qui est définie dans §4.4).

Dans le théorème 6.2.1, nous généralisons ce résultat pour certains feuilletages holomorphes singuliers avec des feuilles partout. Nous expliquons ensuite comment l'espace des feuilles peut être interprété comme sous-espace de $Z_d(X)$. Nous terminons cette partie en utilisant un exemple d'Hirzebruch pour illustrer le théorème 6.1.1.

Dans la dernière partie, nous donnons une solution partielle du deuxième problème pour un type particulier de feuillages.

Nous considérons les feuilletages $\mathbb{F}$ qui sont holomorphes réguliers ou avec singularités et pour lesquels il existe un sous-ensemble ouvert, dense et $\mathbb{F}$-saturé C de X tel que $C/\mathbb{F}$ est un espace complexe. Sous certaines conditions, nous construisons pour ces feuilletages une généralisation de l'espace des feuilles: l'espace des feuilles méromorphes $Z(\mathbb{F})$.

Dans une première étape, nous associons à chaque feuilletage du type ci-dessus une relation d'équivalence méromorphe $M^{\mathbb{F}}$. Nous utilisons ensuite la théorie des relations d'équivalence méromorphes de Grauert et de Siebert pour définir $Z(\mathbb{F})$ comme étant le quotient méromorphe de X par $M^{\mathbb{F}}$. Le théorème de Grauert-Siebert sur les relations d'équivalence méromorphes nous permet de donner une condition qui implique que $Z(\mathbb{F})$ admet une structure complexe canonique (voir le théorème 7.2.5).

Nous donnons d'autres caractérisations de $Z(\mathbb{F})$ pour des cas particuliers. Nous concluons en illustrant la théorie par quelques exemples classiques qui montrent quelques phénomènes qui peuvent apparaître.

Abstract

We study the leaf space $X/\mathbb{F}$ of regular or singular holomorphic foliations $\mathbb{F}$ on a complex manifold X. Using the theory of analytic cycles, we give certain solutions to the two following problems:

First Leaf space problem *Find sufficient conditions which imply that the leaf space of a foliation with leaves everywhere admits a canonical complex structure.*

Second Leaf space Problem *What can be done if $\mathbb{F}$ does not have leaves everywhere, or if it has leaves everywhere but $X/\mathbb{F}$ is not a complex space?*

First we study the First Leaf space problem for regular foliations.

We define the notion of the topological multiplicity $\mu_t(L)$ for some leaves L of the foliation. Then we define a mapping $\zeta_{\mathbb{F}}: G(\mathbb{F}) \longrightarrow Z_d(X)$. Here $G(\mathbb{F})$ is the set of those leaves for which the topological multiplicity is well-defined and $Z_d(X)$ is the space of the analytic cycles of dimension d (d is the dimension of the foliation) with the topology of Barlet. More precisely, $\zeta_{\mathbb{F}}$ associates to each $x \in G(\mathbb{F})$ the cycle $\mu_t(L_x)L_x$. In theorem 6.1.1 we prove the following equivalences:

$X/\mathbb{F}$ is a complex space $\iff$ $G(\mathbb{F}) = X$ and $\zeta_{\mathbb{F}}$ is continuous

$\iff$ There exists a continuous and $\mathbb{F}$-invariant mapping $\varphi: X \longrightarrow Z_d(X)$ such that for each $x \in X$, the support of $\varphi(x)$ is equal to L_x.

$\iff$ The canonical mapping $X \to \widetilde{Z}_d(X)$ that associates to each x the leaf passing through x is continuous ($\widetilde{Z}_d(X)$ is the set of d-dimensional analytic subsets of X; the topology of $\widetilde{Z}_d(X)$ is the Barlet topology defined in §4.4).

In theorem 6.2.1 we generalize this result for certain singular holomorphic foliations that have leaves everywhere. Then we explain how the leaf space can be interpreted as a subspace of $Z_d(X)$. Finally, we use an exemple of Hirzebruch to illustrate theorem 6.1.1.

In the last part we give a partial solution of the second problem for a particular type of foliations.

We consider regular or singular holomorphic foliations for which there exists an open, dense and $\mathbb{F}$-saturated subset C of X such that $C/\mathbb{F}$ is a complex space. Under certain conditions on such foliations we construct a generalisation of the leaf space: the meromorphic leaf space $Z(\mathbb{F})$.

In a first step, we associate a meromorphic equivalence relation $M^{\mathbb{F}}$ to each foliation $\mathbb{F}$ of the above type. Then we use the theory of Grauert and Siebert on meromorphic equivalence relations to define $Z(\mathbb{F})$ as the meromorphic quotient of $M^{\mathbb{F}}$. Using the theorem of Grauert-Siebert on meromorphic equivalence relations, we find a condition which implies that $Z(\mathbb{F})$ has a complex structure (see theorem 7.2.5).

In particular cases, we give other characterisations of $Z(\mathbb{F})$. We conclude with some examples that illustrate some phenomena which can appear.

Contents

Introduction

The notion of differentiable regular foliations on differentiable manifolds X was first studied by Ehresmann, Reeb and Haefliger (see [Ehr51], [Ree52] and [Hae58]). Each such foliation $\mathbb{F}$ induces an equivalence relation $R^{\mathbb{F}}$, for which the classes are exactly the leaves of $\mathbb{F}$. In general, the space of leaves $X/\mathbb{F} := X/R^{\mathbb{F}}$ (called the leaf space) with the quotient topology can be very bad. A compact foliation $\mathbb{F}$ (a foliation with all leaves compact) is called stable if $X/\mathbb{F}$ is Hausdorff, which is equivalent to the condition that each leaf of $\mathbb{F}$ admits a fundamental systems of open saturated neighbourhoods.

To study the stability of compact foliations $\mathbb{F}$, the notion of holonomy group of a leaf was introduced. Reeb in [Ree52] and Epstein in [Eps76] proved that the leaf space $X/\mathbb{F}$ is Hausdorff iff each leaf of $\mathbb{F}$ has a finite holonomy group.

If $\mathbb{F}$ is compact, then the equivalence classes of $R^{\mathbb{F}}$ are compact. Hence there are canonical mappings $\varphi\colon X \longrightarrow \mathcal{K}(X)$ and $\overline{\varphi}\colon X/\mathbb{F} \longrightarrow \mathcal{K}(X)$ given by $\varphi(x) = \overline{\varphi}(L_x) = R^{\mathbb{F}}(x)$, where $\mathcal{K}(X)$ is the set of non-void compact subsets of X with the topology induced by the Hausdorff metric (to define the Hausdorff metric on $\mathcal{K}(X)$, we choose a metric on X; in theorem 4.3.3 we prove that the Hausdorff metric is independent of the choice of the metric on X). We prove in theorem 4.3.8 that

$$\begin{aligned} \mathbb{F} \text{ is stable} &\iff \varphi \text{ is continuous} \\ &\iff \overline{\varphi} \text{ is a homeomorphism onto its image.} \end{aligned}$$

An important class of differentiable foliations are the holomorphic foliations on complex manifolds. For these foliations, Holmann proved in [Hol72] that if the leaf space is Hausdorff, then it admits a canonical complex structure. This proves that for a compact holomorphic foliation $\mathbb{F}$,

$$\mathbb{F} \text{ is stable} \quad\iff\quad X/\mathbb{F} \text{ is a complex space.}$$

The first problem on non-necessary compact holomorphic foliations that we study in this thesis is the

First Leaf space problem *Find sufficient conditions which imply that the leaf space admits a canonical complex structure.*

By the theorem of Holmann, the problem is only topological.

The first idea is to consider the canonical mapping $\psi\colon X \longrightarrow \widetilde{Z}_d(X)$ given by $\psi(x) = L_x$ (where $\widetilde{Z}_d(X)$ is the set of pure d-dimensional analytic subsets of X and $d = \dim \mathbb{F}$). The topology on $\widetilde{Z}_d(X)$ is the topology defined in subsection 4.4. We prove in theorem 6.1.1 that

$$X/\mathbb{F} \text{ is a complex space} \quad \Longleftrightarrow \quad \psi \text{ is continuous.}$$

The set $\widetilde{Z}_d(X)$ does not admit a complex structure. In [Bar75], Barlet defined a complex space $\mathcal{B}_d(X)$, of which the underlying set $C_d(X)$ is the set of compact pure d-dimensional analytic cycles of X (a cycle of X is an analytic subset of X with multiplicities). In [Sie92], Siebert studied the set $Z_d(X)$ of d-dimensional analytic cycles introduced by Barlet. The topology of $Z_d(X)$ is the so-called Barlet-topology defined in subsection 2.1. The topology on $C_d(X)$ given by $\mathcal{B}_d(X)$ is finer than the topology on $C_d(X)$ as subspace of $Z_d(X)$.

The second idea of this thesis is to try to associate to each holomorphic foliation $\mathbb{F}$ a mapping $X \to Z_d(X)$, that helps us to study the leaf space of $\mathbb{F}$.

For certain leaves L of $\mathbb{F}$, we define the topological multiplicity $\mu_t(L)$ of this leaf (compare definition 5.2.8). The good set $G(\mathbb{F})$ of $\mathbb{F}$ consists of the points x for which the topological multiplicity of L_x is well-defined. If $\mathbb{F}$ is compact, then the topological multiplicity of a leaf coincides with the order of the holonomy group of this leaf (see theorem 5.2.12), and G is nothing else but the good set of $\mathbb{F}$ (compare [Hol78]). The mapping $\zeta_{\mathbb{F}}\colon G \longrightarrow Z_d(X)$ that associates to each point $x \in G$ the cycle $\mu_t(L_x)[L_x]$ is well-defined.

The question is if this mapping is continuous or where it is continuous. A first part of the answer is given by theorem 5.4.5 that explains the relation between the points where $\zeta_{\mathbb{F}}$ is continuous and the points where $X/\mathbb{F}$ is Hausdorff. Another part of the answer is given by theorem 6.1.1 which says that

$$X/\mathbb{F} \textit{ is a complex space} \quad \Longleftrightarrow \quad G = X \textit{ and } \zeta_{\mathbb{F}} \textit{ is continuous.}$$

The two principal properties of $\zeta_{\mathbb{F}}$ are that it is $\mathbb{F}$-invariant and that $|\zeta_{\mathbb{F}}(x)| = L_x$ for each $x \in G$. The existence of such a mapping is sufficient (compare theorem 6.1.1), i.e.

$$X/\mathbb{F} \textit{ is a complex space} \quad \overset{(*)}{\Longleftrightarrow} \quad \begin{cases} \textit{There exists a continuous and } \mathbb{F}\textit{-invariant} \\ \textit{mapping } \varphi\colon X \longrightarrow Z_d(X) \textit{ such that } |\varphi(x)| = \\ L_x \textit{ for each } x \in X \end{cases}$$

If φ is a mapping with the properties of the right side of the equivalence $(*)$, then the induced mapping $\overline{\varphi}\colon X/\mathbb{F} \longrightarrow Z_d(X)$ is a homeomorphism onto its image and an analytic family (compare theorem 6.3.1).

In [BB72], Baum and Bott introduced the notion of singular holomorphic foliations, since there are many manifolds that cannot be foliated by a non-trivial regular foliation. The singular holomorphic foliations were systematically studied by Bohnhorst and Reiffen in [BR85] and in [Rei97].

A (representative of a) singular holomorphic foliation $\mathbb{F}$ of dimension d on a complex manifold X can be defined as a regular holomorphic foliation $\mathbb{F}'$ of dimension d on $X \setminus A$, where $A \subset X$ is an analytic subset of X of codimension ≥ 2. Equivalently, it can be defined by certain coherent involutive sheaves of vector fields or Pfaffian forms on X (see subsection 3.2, [Rei97] or [BR85]).

As in the regular case, it is possible to define the notion of leaves of a singular holomorphic foliation (compare [HKR99] or [Rei97]). The problem is that there exist singular holomorphic foliations that do not have leaves everywhere. For example, consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $(\lambda, z) \to (\lambda z_1, \lambda z_2)$. It defines a singular holomorphic foliation $\mathbb{F}$ on $\mathbb{C}^2$ (each action of a complex Lie group on a complex manifold defines a singular holomorphic foliation on this manifold, as it is shown by [Rei97, Example 3.13(6)]). The leaves of $\mathbb{F}$ are exactly the orbits of the form $\{\lambda x \mid \lambda \in \mathbb{C}^*\}$, where $x \neq 0$. The foliation does not have leaves at 0 (compare example 7.4.1).

The third idea of this thesis is to solve the First Leaf space problem for singular holomorphic foliations with the tools developed above. If the canonical mapping $X \to \widetilde{Z}_d(X)$, $x \mapsto L_x$, is continuous, then $X/\mathbb{F}$ is Hausdorff. But, since there is no analog of the theorem of Holmann on leaf spaces, we cannot conclude that $X/\mathbb{F}$ is a complex space. Hence we consider the leaves with multiplicities. The equivalence $(*)$ is also true for singular holomorphic foliations with certain properties (compare theorem 6.2.1). For the proof, we use the theorem of Grauert on semi-proper equivalence relations (compare [Kau93] or theorem 6.2.4).

In the last part of this thesis, we search a solution of the

Second Leaf space Problem *What can be done if $\mathbb{F}$ does not have leaves everywhere, or if it has leaves everywhere but $X/\mathbb{F}$ is not a complex space?*

When $X/\mathbb{F}$ is not a complex space, two types of problems can appear: the leaves cannot be separated, but there is no problem of multiplicity (as in example 6.1.3), or there is a problem with the multiplicity (as in the example developed by Müller in [Mue86]). We can find a solution of the Second Leaf space Problem for certain singular holomorphic foliations for which there is no problem of multiplicity, even if they do not have leaves everywhere: we find a complex space that is a generalisation in a certain sense of the leaf space; it parametrizes the foliation almost everywhere.

The above example illustrates this fact: $\mathbb{F}$ is the foliation on $\mathbb{C}^2$ given by the action $(\lambda, z) \mapsto (\lambda z_1, \lambda z_2)$ of $\mathbb{C}^*$. If X' is the blow-up of $\mathbb{C}^2$ at 0, and $\sigma : X' \longrightarrow X$ is the blowing-down mapping, then consider the foliation $\mathbb{F}'$ on X' that is the lifting of $\mathbb{F}$ to σ. It is a regular foliation, and $X'/\mathbb{F}'$ is a complex space isomorphic to $\mathbb{P}_1$ (the complex projective space of dimension 1).

We can generalize the above condition for certain singular holomorphic foliations. We consider singular or regular holomorphic foliations for which there exists an open, dense and $\mathbb{F}$-saturated subset C of X such that $C/\mathbb{F}|_C$ is a complex space. Under certain conditions ($\mathbb{F}$ is M-analytic and separable in a certain sense; see definition 7.1.9 and definition 7.2.4), we find a subset $Z(\mathbb{F})$ of $Z_d(X)$ such that $C/\mathbb{F}$ is a dense subset of $Z(\mathbb{F})$, a proper modification $\sigma : X' \longrightarrow X$ and an equivalence relation R' on X' such that X'/R' is a complex space and homeomorph to $Z(\mathbb{F})$ (compare theorem 7.2.5). The space $Z(\mathbb{F})$ is called the meromorphic leaf space of $\mathbb{F}$.

The main tools of the proof are the theorems of Grauert and Siebert on meromorphic equivalence relations (compare [Gra86], [Sie92] or definition 2.6.1).

Since the theory of analytic cycles is developed for complex spaces, possibly theorems of the same type as theorem 6.1.1 or theorem 6.2.1 can be find to solve the First Leaf space problem for singular holomorphic foliations on a normal complex space or even on a maximal complex space.

If X is compact, then $Z(\mathbb{F})$ is separable in the sense of definition 7.2.4. Perhaps the last part of this thesis gives a new way to study the conjecture of Holmann saying that for each compact regular holomorphic foliation $\mathbb{F}$ on a compact manifold X, the leaf space $X/\mathbb{F}$ is a complex space.

Complex analysis, theory of cycles and theory of foliations

This Part restates known results. A first section is dedicated to the complex analysis, a second to the theory of cycles and a third to the theory of foliations.

Most of the notions and notations used in this thesis are presented. This helps the reader, because he can refer directly to this part instead of different articles.

1 Complex analysis

This section restates some known results of complex analysis. The proofs are in general not present, but the reader can find them in the literature. Some articles or books are cited in the text.

1.1 Preliminaries

In this subsection, we present standard notions and fix the notations.

In this thesis, all complex spaces are reduced (see [KK83], [Fis76] or [GR84]). All complex spaces or manifolds are paracompact. If X is a complex space, we note in general its structure sheaf by $\mathcal{O}$, or ${}_X\mathcal{O}$ if the space must be specified. The sheaf of continuous functions on X will be denoted by $\mathcal{C}$ (or ${}_X\mathcal{C}$). We denote the set of holomorphic, resp. continuous, functions on an open subset $U \Subset X$[1] by $\mathcal{O}(U)$, resp. $\mathcal{C}(U)$.

If X is a topological space, $x \in X$ and $A \subset X$, then we denote by A_x the ***germ of the subset A at x***, i.e. A_x is the equivalence class of the equivalence relation $\underset{x}{\sim}$ given by

$$A_1 \underset{x}{\sim} A_2 \quad :\Longleftrightarrow \quad \text{There exists an open neighbourhood } U \Subset X \text{ of } x \text{ such that } A_1 \cap U = A_2 \cap U.$$

[1]This notation means that U is open in X

An equivalence relation R on a space X is uniquely characterized by its graph $R \subset X \times X$ given by

$$R = \{(x, y) \in X \times X \mid xRy\}.$$

Let $p_j : R \longrightarrow X$ be the projection to the jth factor. If $A \subset X$, then we define the ***R-saturated hull*** $R(A)$ of A by

$$R(A) := p_1(p_2^{-1}(A)) = \{x \in X \mid \exists y \in A \text{ such that } xRy\}.$$

If $x \in X$, then $R(x)$ is the equivalence class of x. For $A \subset X$, we denote $R|_A := R \cap (A \times A)$, being the restriction of R on A. The canonical projection onto the quotient is denoted by $\pi : X \longrightarrow X/R$. Hence, $A/R|_A = \pi(A)$ and $R(A) = \pi^{-1}(\pi(A))$.

A subset A of X is called ***R-saturated*** if $A = R(A)$ and a function $f \in \mathcal{O}(U)$, where U is an open R-saturated subset of X, is called ***R-invariant*** if $f(x) = f(y)$ for all $x \in X$ and $y \in R(x)$.

1.1.1 Definition An equivalence relation R on a complex space X is called ***analytic*** if its graph is an analytic subset of $X \times X$.

If X is a topological space, we impose the ***quotient-topology*** on X/R, i.e. $U \subset X/R$ is open in X/R iff $\pi^{-1}(U)$ is open in X.

If X is a complex space, we impose a structure sheaf $\mathcal{Q}$ on X/R making $(X/R, \mathcal{Q})$ a ringed space: the sheaf $\mathcal{Q}$ is given by

$$\mathcal{Q}(V) := \{f : V \longrightarrow \mathbb{C} \text{ continuous} \mid f \circ \pi \in \mathcal{O}(\pi^{-1}(V))\}$$

for each $V \Subset X/R$.

An equivalence relation R on X is called ***open*** if for each $U \Subset X$, the saturated hull $R(U)$ of U is open in X.

1.1.2 Lemma *For an equivalence relation R on a locally compact metric space X, the following conditions are equivalent:*

(a) *R is open.*

(b) *The canonical projection $\pi : X \longrightarrow X/R$ is open.*

(c) *For each $x \in X$ and for each sequence (x_k) such that $x_k \to x$ and for each $y \in R(x)$, there exists a sequence (y_k) such that $y_k \in R(x_k)$ and $y_k \to y$.*

(d) *For each R-saturated subset A of X, $\mathring{A}$ is R-saturated.*

(e) *For each R-saturated subset A of X, $\overline{A}$ is R-saturated.*

An equivalence relation R on a hausdorff topological space X is called ***proper*** if for each $K \subset X$ compact, the saturated hull $R(K)$ of K is compact. An equivalence relation R is called ***quasi-finite*** if $R(x)$ has a finite number of elements for each $x \in X$. If R is quasi-finite and proper, then it is called ***finite***.

A holomorphic mapping is called ***finite*** if it is proper and discrete. A holomorphic mapping is called ***quasi-finite*** if each fibre has a finite number of elements.

1.1.3 Definition A local analytic subset $A \subset X$ of a complex space X is called ***thin*** if it is nowhere dense in X. A subset $S \subset X$ is called ***analytically thin*** if for each point $x \in \overline{S}$ there exists a local analytic subset $A \subset X$ such that $A_x \supset \overline{S}_x$ and A is thin in X.

A proper surjective holomorphic mapping $f : X \longrightarrow Y$ is called a ***(proper) modification*** if there exist thin analytic subsets $A \subset X$ and $B \subset Y$ such that $B = f(A)$ and $f|_{X \setminus A} : X \setminus A \longrightarrow Y \setminus B$ is biholomorphic (see for example [Pet94] or [Fis76]).

A complex space X is called ***normal*** if the equation $\mathcal{O} = \widetilde{\mathcal{O}}$ holds, where $\widetilde{\mathcal{O}}$ is the sheaf of weakly holomorphic functions[2] on X. A complex space X is called ***maximal*** if the equation $\mathcal{O} = \widetilde{\mathcal{O}} \cap \mathcal{C}$ holds. Clearly, a normal complex space is maximal. Note that a complex space is normal iff it is maximal and locally irreducible.

A finite holomorphic mapping $f : \widetilde{X} \longrightarrow X$ is called a ***normalization*** of X if $\widetilde{X}$ is normal and f is a modification. The ***maximalization*** $\widehat{X}$ of a complex space $X = (X, \mathcal{O})$ is the complex space $\widehat{X} := (X, \widehat{\mathcal{O}})$, where $\widehat{\mathcal{O}} = \widetilde{\mathcal{O}} \cap \mathcal{C}$.

A ***meromorphic mapping*** (in the sense of Remmert [Rem57]) $f : X \longrightarrow Y$ is an analytic subset Γ_f of $X \times Y$ such that the projection $\pi_X : \Gamma_f \longrightarrow X$ on X is a proper modification.

1.2 Generically open mappings

A continuous mapping $f : S \longrightarrow T$ from a topological space S to a topological space T is called ***open*** if for each $U \Subset S$, $f(U)$ is open in T. In the context of holomorphic mappings, this notion is generalized by the notion of generically open mappings.

1.2.1 Definition A holomorphic mapping $f : X \longrightarrow Y$ is called ***generically open*** if the image of any irreducible component of X contains a non-void open subset of Y.

1.2.2 Lemma *A generically open holomorphic mapping $f : X \longrightarrow Y$ has the following properties:*

(a) *If $N \subset Y$ is analytic and thin in Y then $f^{-1}(N)$ is thin in X.*

(b) *If X_ν is an irreducible component of X, then there exists exactly one irreducible component Y_ν of Y such that $f(X_\nu) \subset Y_\nu$.*

[2]A function $f \in \mathcal{O}(U \setminus A)$, where U is open in X and A thin and analytic in U, is called **weakly holomorphic on** U if f is locally bounded at A

Proof For (a), suppose that $f^{-1}(N)$ is not thin, i.e. there exists $U \Subset X$ such that $U \subset f^{-1}(N)$. Then, by the identity theorem, there exists an irreducible component X_ν of X such that $X_\nu \subset f^{-1}(N)$. Thus $f(X_\nu) \subset N$ and contains a non-void open subset of Y, which contradicts the fact that N is thin in X.
For (b), let X_ν be an irreducible component of X and Y_ν be an irreducible component of Y such that $f(X_\nu) \cap Y_\nu \neq \emptyset$. Thus $f^{-1}(Y_\nu) \cap X_\nu$ is analytic and not nowhere dense in X_ν. Then $f^{-1}(Y_\nu) \cap X_\nu = X_\nu$ which implies that $f(X_\nu) \subset Y_\nu$.
For the uniqueness, suppose that $f(X_\nu) \subset Y_\nu \cap Y'_\nu$. Thus $f(X_\nu) \subset \operatorname{Sing} Y$, which is in contradiction with the fact that f is generically open, because $\operatorname{Sing} Y$ is thin in Y. □

1.2.3 Proposition *An holomorphic mapping $f : X \longrightarrow Y$ is generically open iff there exists a thin analytic subset $A \subset X$, such that $f|_{X \smallsetminus A}$ is open.*

Proof *if* is clear. We have to prove *only if*. For that we define

$$E_\nu := \{x \in X_\nu \mid \dim_x f^{-1}(f(x)) > \dim X_\nu - \dim Y_\nu\},$$

for each irreducible component X_ν of X (Y_ν denotes the irreducible component of Y such that $f(X_\nu) \subset Y_\nu$). We define $E := \bigcup_\nu E_\nu$. By [Fis76, 3.6], E is analytic in X and E_ν is analytic in X_ν. In [Sie93, Lemma 1.1], Siebert proved that E is thin in X and E_ν is thin in X_ν. We set $A := E \cup f^{-1}(N)$, where N is the non-normal locus of Y. The set A is analytic and thin in X by lemma 1.2.2. By [Fis76, 3.10], $f|_{X \smallsetminus A}$ is open. □

1.2.4 Lemma *Let $f : X \longrightarrow Y$ be a generically open holomorphic mapping. If $N \subset Y$ is analytically thin in Y, then $f^{-1}(N)$ is analytically thin in X.*

Proof Let $x \in N$, let $U \Subset Y$ be a neighbourhood of x and let $A \subset U$ be analytic thin in U such that $\overline{N} \subset A \subset U$. By proposition 1.2.3, $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is generically open. Hence by lemma 1.2.2, $f^{-1}(A)$ is analytic and thin in $f^{-1}(U)$ which concludes the proof. □

1.2.5 Definition For a holomorphic mapping $f : X \longrightarrow Y$, the ***singular locus*** of f is given by

$$\operatorname{Sing} f := \operatorname{Sing} f^*({}_Y\Omega).$$

where $f^*{}_Y\Omega$ denotes the ${}_X\mathcal{O}$-module generated by the Pfaffian forms $f^*\omega$, $\omega \in {}_Y\Omega$. (See [Fis76, 2.14] for a description of the singular locus of a sheaf of modules).

By [Fis76, 2.14], this set is always analytic and thin. Furthermore, by [Rei97, 3.18], if X and Y are manifolds, then

$$X \smallsetminus \operatorname{Sing} f = \{x \in X \mid d_x f \text{ has maximal rank}\}$$

1.2.6 Definition If $f : X \longrightarrow Y$ is a generically open holomorphic mapping, then

$$\operatorname{Sg} f := \operatorname{Sing} X \cup f^{-1}(\operatorname{Sing} Y) \cup \operatorname{Sing} f.$$

This is a thin analytic subset of X by [BR85, §4].

1.2.7 Lemma *The equation*

$$X \setminus \operatorname{Sg} f = \{x \in X \setminus \operatorname{Sing} X \mid f(x) \notin \operatorname{Sing} Y \text{ and } d_x f \text{ has maximal rank}\}$$

holds. □

This implies that $f|_{X \setminus Sgf} : X \setminus \operatorname{Sg} f \longrightarrow Y \setminus \operatorname{Sing} Y$ is a submersion.

1.3 Analytic coverings

In this subsection, we present the notion of analytic coverings. This is a generalisation of the well-known notion of coverings. This generalisation was introduced for the first time in [GR58] (see also [GR84] or [DG94]).

1.3.1 Definition A holomorphic mapping $\eta : X \longrightarrow Y$ between two complex spaces is called an ***analytic covering*** if η is finite and surjective and if there exists a thin analytic subset $T \subset Y$ such that

(a) $\eta^{-1}(T)$ is a thin analytic subset of X

(b) The induced mapping $X \setminus \eta^{-1}(T) \to Y \setminus T$ is locally biholomorphic.

The minimal T satisfying the conditions (a) and (b) is called the ***critical locus*** of the covering and the set B of all points of X where η is not locally biholomorphic is called the ***branch locus*** of the covering. The covering is called ***unbranched*** if B is empty.

1.3.2 Proposition *An analytic covering $\eta : X \longrightarrow Y$ with critical locus T has the following properties:*

(a) *For each open set $V \Subset Y$, the induced mapping $\eta^{-1}(V) \to V$ is an analytic covering with critical locus $V \cap T$.*

(b) *For each analytically thin set $M \subset X$, resp. $N \subset Y$, the set $\eta(M)$, resp $\eta^{-1}(N)$, is analytically thin in X, resp. in Y.*

(c) *If Y is locally pure dimensional, then X is locally pure dimensional.*

(d) *If Y is locally irreducible, then η is open.*

(e) *The function $y \mapsto \operatorname{Card}(\eta^{-1}(y))$ is locally constant on $Y \setminus T$.*

For the proof see for example [GR84, 7.2.1].

1.3.3 Definition An analytic covering $\eta : X \longrightarrow Y$ with critical locus T is called ***sheeted*** if there exists a number b such that $\operatorname{Card}(\eta^{-1}(y)) = \text{const} = b$ for all $y \in Y \setminus T$.

1.3.4 Lemma *If Y is irreducible, then η is sheeted.*

Proof By [GR84, 7.2.1], the number of sheets of η is constant on $Y \setminus (\operatorname{Sing} Y \cup T)$, where T is the critical locus of η. Since the number of sheet is locally constant on $Y \setminus T$, it is constant on $Y \setminus T$. Hence η is sheeted. $\square$

In general, an analytic covering is not open as it is shown by the following example:

1.3.5 Example If X is a not locally irreducible complex space then denote by $\nu : \widetilde{X} \longrightarrow X$ its normalization. The mapping ν is a one-sheeted covering, but it is not open.

An important class of analytic coverings is given by

1.3.6 Theorem *Every open finite holomorphic surjection $\eta : X \longrightarrow Y$ is an analytic covering.*

For the proof see [GR84, 7.2.3].

One characterizes analytic coverings in the following way:

1.3.7 Proposition *For a finite holomorphic surjection $\eta : X \longrightarrow Y$, the following conditions are equivalent*

(a) *η is an analytic covering*

(b) *η is generically open*

(c) *The image by η of any irreducible component of X is an irreducible component of Y.*

For the proof see [GR84, 9.3.3].

2 Cycles, geometric flatness and meromorphic equivalence relations

In this section, we collect many results on cycles and on notions linked to the cycles, as geometrically flatness, fibre-cycle space or meromorphic equivalence relations. The most important references are [Bar75], [Sie92] and [Sie94].

2.1 Cycles and the Barlet-topology

This subsection introduces the notion of analytic cycles and the Barlet-topology, a useful topology on the set of analytic cycles. At the end, we present some useful constructions.

In this subsection, X denotes a complex space.

2.1.1 Definition An ***analytic cycle*** Z of a complex space X is a locally finite formal sum

$$Z = \sum_{k \in I} n_k Z_k$$

of irreducible analytic subsets $Z_k \neq \emptyset$ of X, with coefficients $n_k \in \mathbb{N}_{>0}$.
If $I = \emptyset$, then $Z = [\emptyset]$ is called the ***null cycle***.
The ***support*** $|Z|$ of Z is the underlying analytic subset $\bigcup_{k \in I} Z_k$ of X.
A cycle Z is called ***d-dimensional*** if Z_k is d-dimensional for all $k \in I$. A cycle is called ***reduced*** if $n_k = 1$ for all $k \in I$.

If S is an analytic subset of Y, and if $S = \bigcup_{k \in I} S_k$ is its decomposition into irreducible components, then we denote the reduced cycle $\sum_{k \in I} 1 \cdot S_k$ by $[S]$ to distinguish clearly between the cycle $[S]$ and the analytic set S.

The set of pure d-dimensional cycles of X is denoted by $Z_d(X)$.

In [Bar75], Barlet has imposed a topology on $Z_d(X)$, which we call the ***Barlet-topology***. In the following, we explain the construction of a subbase of neighbourhoods of that topology.

2.1.2 Definition A (d-dimensional) ***scale*** $\mathcal{S} := (\chi : V \longrightarrow \Omega, W, D)$ is a triple consisting of

- a closed embedding $\chi : V \longrightarrow \Omega$ of an open subset $V \Subset X$ into a domain $\Omega \Subset \mathbb{C}^N$
- open polycylinders $D \Subset \mathbb{C}^d$ and $W \Subset \mathbb{C}^{N-d}$ such that the following condition is satisfied:

$$\overline{W} \times \overline{D} \subset \Omega. \tag{S_1}$$

The open subset $|\mathcal{S}| := \chi^{-1}(W \times D)$ of V is called the ***support*** of the scale.

2.1.3 Definition A d-dimensional scale $\mathcal{S} := (\chi : V \longrightarrow \Omega, W, D)$ is called ***adapted*** to some cycle $Z \in Z_d(X)$ if the following condition is satisfied:

$$\chi^{-1}(\partial W \times \overline{D}) \cap |Z| = \emptyset. \tag{S_2}$$

Figure 1 illustrates the situation.

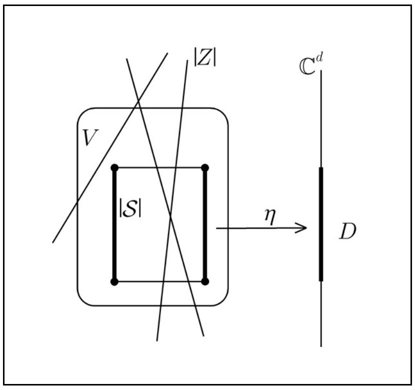

Figure 1: A scale $\mathcal{S}$ adapted to a cycle Z

2.1.4 Lemma *If $\mathcal{S} := (\chi : V \longrightarrow \Omega, W, D)$ is a scale adapted to a cycle Z, then the mapping $\eta : |Z| \cap |\mathcal{S}| \longrightarrow D$ given by $\eta := (\mathrm{pr}_2 \circ \chi)|_{|Z| \cap |\mathcal{S}|}$ is an open analytic covering.*

Proof First, we prove that η is proper: If $K \subset D$ is compact, then $\overline{W} \times K \subset \mathbb{C}^N$ is compact. Hence $\chi^{-1}(\overline{W} \times K) \cap |Z| \subset V$ is compact too. But, using property (S_2),

$$\begin{aligned} \chi^{-1}(\overline{W} \times K) \cap |Z| &= (\chi^{-1}(W \times K) \cap |Z|) \cap \underbrace{(\chi^{-1}(\partial W \times K) \cap |Z|)}_{=\emptyset} \\ &= \chi^{-1}(W \times K) \cap |Z| \\ &= \eta^{-1}(K), \end{aligned}$$

which proves that $\eta^{-1}(K)$ is compact.
Since $|\mathcal{S}|$ is a Stein Space (i.e. holomorphically separable[3] and holomorphically convex, as it is defined for example in [KK83, section 51]), $|\mathcal{S}| \cap |Z|$ is also a Stein space. Hence $\eta^{-1}(t)$ is also a Stein space for each $t \in D$. Let $t \in D$ and let C be a connected component of $\eta^{-1}(t)$. The set C is a Stein space and so is holomorphically separable (This argumentation use theorems that we can find in [KK83, section 51]). Since C is compact, each holomorphic function on C is constant (maximum principle). Thus C must be one point. It follows that $\eta^{-1}(t)$ is discrete, and then η is finite.
Since D is locally irreducible and $\dim |Z| = \dim D$, we use [Fis76, 3.10] to conclude that η is open.
Since η is finite and open, it is an analytic covering by theorem 1.3.6. □

[3]A ringed space $(X, \mathcal{O})$ is called **holomorphically separable** if for $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists a function $f \in \mathcal{O}(X)$ such that $f(x_1) \neq f(x_2)$.

2.1.5 Definition Let $\mathcal{S} := (\chi : V \longrightarrow \Omega, W, D)$ be a scale adapted to a cycle $Z = \sum_{k \in I} n_k Z_k$. The ***degree*** $\deg_{\mathcal{S}}(Z)$ of Z relatively to $\mathcal{S}$ is defined by

$$\deg_{\mathcal{S}}(Z) := \sum_{k \in I} n_k \nu_k$$

where ν_k is the number of sheets of the analytic covering $\eta : Z_k \cap |\mathcal{S}| \longrightarrow D$ ($\nu_k := 0$ if $Z_k \cap |\mathcal{S}| = \emptyset$). This number is well-defined by lemma 1.3.4, since Z_k is irreducible.

2.1.6 Remark For each scale $\mathcal{S}$, $\deg_{\mathcal{S}}([\emptyset]) = 0$.

The set

$$B_{\mathcal{S}}(k) := \{Z \in Z_d(X) \mid \mathcal{S} \text{ is adapted to } Z \text{ and } \deg_{\mathcal{S}} Z = k\},$$

where $\mathcal{S}$ is a d-dimensional scale and $k \in \mathbb{N}$, is called a ***scale neighbourhood*** of all its elements. The set of all scale neighbourhoods forms a subbase of the ***Barlet-topology*** on $Z_d(X)$. Note that 0 is a possible value for k.

2.1.7 Proposition *$Z_d(X)$ with the Barlet-topology has the following properties:*

(a) *it is Hausdorff*

(b) *it is first countable, i.e. for each $Z \in Z_d(X)$, there exists a countable base of neighbourhoods of Z.*

Proof In order to prove the property of separation let Z and Z' be two different cycles in $Z_d(X)$. We write $Z = \sum_{k \in I} n_k Z_k$ and $Z' = \sum_{k \in I'} n'_k Z'_k$. There are two possibilities: $|Z| \neq |Z'|$ or $|Z| = |Z'|$.
In the first case we choose a scale $\mathcal{S}$ adapted to Z with $\deg_{\mathcal{S}} Z \neq 0$ such that $|Z'| \cap \overline{|\mathcal{S}|} = \emptyset$. Then $|Z| \in B_{\mathcal{S}}(\deg_{\mathcal{S}} Z)$ and $Z' \in B_{\mathcal{S}}(0)$ and $B_{\mathcal{S}}(\deg_{\mathcal{S}} Z) \cap B_{\mathcal{S}}(0) = \emptyset$.
In the second case there exists an irreducible component S of $|Z| = |Z'|$ such that $\mu \neq \mu'$ where μ, resp. μ', is the coefficient of S in Z, resp. in Z'. We choose a scale $\mathcal{S}$ adapted to Z such that $S \cap |\mathcal{S}| = |Z| \cap |\mathcal{S}| \neq \emptyset$. Then $B_{\mathcal{S}}(\deg_{\mathcal{S}} Z) \cap B_{\mathcal{S}}(\deg_{\mathcal{S}} Z') = \emptyset$ because $\deg_{\mathcal{S}} Z \neq \deg_{\mathcal{S}} Z'$.
For the proof of the first countable property, see [Sie94, page 245]. □

By proposition 2.1.7, it is possible to test for compactness or closedness of subsets of $Z_d(X)$ using sequences of cycles. For a better comprehension of the convergence of sequences in $Z_d(X)$, compare proposition 2.1.11.

2.1.8 Example Let (Z_k) be a sequence of cycles in $Z_d(X)$ such that $|Z_k| \to \partial X$ in the following sense: for each compact subset $K \subset X$, there exists k_0 such that $|Z_k| \cap K = \emptyset$ for each $k \geq k_0$. A fundamental systems of open neighbourhoods of the null cycle $[\emptyset]$ is given by the open sets of the form $\bigcap_{j=1}^{N} B_{\mathcal{S}_j}(0)$ with N scales $\mathcal{S}_1, \ldots, \mathcal{S}_N$. Since $|\mathcal{S}_1| \cup \cdots \cup |\mathcal{S}_N|$ is relatively compact in X, there exists k_0 such that $|Z_k| \cap \overline{|\mathcal{S}_j|} = \emptyset$ for each $k \geq k_0$ and each $j = 1, \ldots, N$. Thus $Z_k \in \bigcap_{j=1}^{N} B_{\mathcal{S}_j}(0)$ for each $k \geq k_0$. This proves that $Z_k \to [\emptyset]$.

2.1.9 Definition A sequence (A_k) of non-void closed subsets of X ***converges set theoretically*** to a closed set A if

(a) A is the set of all accumulation points of all sequences (a_k) with $a_k \in A_k$ for all k.

(b) For each open neighbourhood $U \Subset X$ of A and for each compact $K \subset X$, $A_k \cap K \subset U$ for large k.

We write $A = \lim_{k\to\infty} A_k$.

2.1.10 Remark A subbase for the topology on the set of closed subsets of X that defines this convergence is given by the sets $V(K, U) := \{A \subset X \text{ closed} \mid A \cap K \subset U\}$ and $W(K) := \{A \subset X \text{ closed} \mid A \cap K = \emptyset\}$ where $K \subset X$, resp. $U \subset X$, is an arbitrary compact, resp. open, subset of X.

2.1.11 Proposition *If* $Z = \lim_{k\to\infty} Z_k$ *in* $Z_d(X)$, *then* $|Z| = \lim_{k\to\infty} |Z_k|$.

Proof Let

$$L := \{x \in X \mid x \text{ is an accumulation point of a sequence } (x_k) \text{ with } x_k \in |Z_k| \text{ for all } k\}.$$

We prove that $|Z| \subset L$. Let $x \in |Z|$. For each scale $\mathcal{S}$ adapted to Z with $\deg_{\mathcal{S}} Z \neq 0$ such that $x \in |\mathcal{S}|$, there exists k_0 such that $\mathcal{S}$ is adapted to Z_k and $\deg_{\mathcal{S}} Z_k = \deg_{\mathcal{S}} Z$ for all $k \geq k_0$. Then there exists a sequence (x_k), $x_k \in |Z_k|$ for all k, such that x is an accumulation point of (x_k).
We prove that $L \subset |Z|$. Let $x \in L$ and let (x_k) such that $x_{k_l} \to x$. Suppose that $x \notin |Z|$. There exists a scale $\mathcal{S}$ such that $\overline{|\mathcal{S}|} \cap |Z| = \emptyset$ and $x \in |\mathcal{S}|$. This scale is adapted to Z. Thus $\deg_{\mathcal{S}} Z = 0$. Since $Z_k \to Z$, $Z_k \in B_{\mathcal{S}}(0)$ for all $k \geq k_0$, i.e. $|Z_k| \cap \overline{|\mathcal{S}|} = \emptyset$. Then $x_{k_l} \not\to x$, which is a contradiction.
The first part proves that $|Z| = L$. Let U be an open neighbourhood of $|Z|$ in X and $K \subset X$ be compact. To complete the proof, we have to see that there exists k_0 such that $|Z_k| \cap K \subset U$ for each $k \geq k_0$. The set $\widetilde{K} := (X \setminus U) \cap K$ is a compact subset of X. Hence, there exists N d-dimensional scales $\mathcal{S}_1, \ldots, \mathcal{S}_N$ adapted to Z such that $|\mathcal{S}_j| \cap |Z| = \emptyset$ for each j and $\widetilde{K} \subset \bigcup_{j=1}^N |\mathcal{S}_j|$. Hence, for each j, $\deg_{\mathcal{S}_j}(Z) = 0$. Since $Z_k \to Z$ and $\bigcap_{j=1}^N B_{\mathcal{S}_j}(0)$ is an open neighbourhood of Z, there exists k_0 such that $Z_k \in \bigcap_{j=1}^N B_{\mathcal{S}_j}(0)$ for each $k \geq k_0$. Hence, for each $k \geq k_0$, $\deg_{\mathcal{S}_j}(Z_k) = 0$ and thus $|Z_k| \cap \left(\bigcup_{j=1}^N |\mathcal{S}_j|\right) = \emptyset$. Hence $|Z_k| \cap \widetilde{K} = \emptyset$. Finally

$$|Z_k| \cap K = (\underbrace{|Z_k| \cap \widetilde{K}}_{=\emptyset}) \cup (\underbrace{|Z_k| \cap K \cap U}_{\subset U}) \subset U,$$

which completes the proof. ◻

The set of all analytic cycles on X is denoted by $Z_*(X)$. On this set we impose the topology induced by all projections

$$Z_*(X) \longrightarrow Z_d(X), \quad Z \mapsto Z^{(d)}$$

where $Z^{(d)} = \sum_{\dim Z_k = d} n_k Z_k$ denotes the pure d-dimensional part of the cycle $Z = \sum_{k\in I} n_k Z_k$. A subbase of this topology in given by the sets

$$B_{\mathcal{S}}(k) := \{Z \in Z_*(X) \mid \mathcal{S} \text{ is adapted to } Z^{(d)} \text{ and } \deg_{\mathcal{S}} Z^{(d)} = k\}$$

where $\mathcal{S}$ is a d-dimensional scale and $k \in \mathbb{N}$. This topology is called the ***Barlet-topology*** on $Z_*(X)$. This topology is of course Hausdorff and first countable.

We sometimes use the notation $B_{\mathcal{S}} := \bigcup_{k\geq 0} B_{\mathcal{S}}(k)$.

An important class of open subsets of $Z_*(X)$ is given by the following:

2.1.12 Lemma *For $U \Subset X$, the set*

$$\{Z \in Z_*(X) \mid |Z| \cap U \neq \emptyset\} \subset Z_*(X)$$

is open in $Z_*(X)$.

Proof Let $Z \in Z_*(X)$ such that $|Z| \cap U \neq \emptyset$. If $\mathcal{S}$ is a d-dimensional scale associated to $Z^{(d)}$ such that $|Z^{(d)}| \cap |\mathcal{S}| \neq \emptyset$ and $|\mathcal{S}| \subset U$, then $k := \deg_{\mathcal{S}}(Z^{(d)}) \neq 0$. By construction, $B_{\mathcal{S}}(k)$ is a neighbourhood of Z, and for each $Z' \in B_{\mathcal{S}}(k)$, $|Z'| \cap U \neq \emptyset$. Hence

$$B_{\mathcal{S}}(k) \subset \{Z \in Z_*(X) \mid |Z| \cap U \neq \emptyset\},$$

which concludes the proof. $\square$

2.1.13 Definition Let $f : X \longrightarrow Y$ be holomorphic and let $Z = \sum_k n_k Z_k \in Z_*(X)$ such that $f|_{|Z|}$ is finite. Then $f|_{Z_k} : Z_k \longrightarrow f(Z_k)$ is a μ_k-sheeted analytic covering. We set

$$f_*(Z) := \sum_k (n_k \cdot \mu_k)\, f(Z_k) \ \in Z_*(Y).$$

Barlet proves that f_* is continuous in particular situations (see [Bar75, Theorem 6]).

The Barlet-topology behaves well under decomposition of X into irreducible components X_ν. Let $i_\nu : X_\nu \longrightarrow X$ be the natural inclusions and $i : \coprod_\nu X_\nu \longrightarrow X$ be given by $i(x) := i_\nu(x)$ if $x \in X_\nu$. Then

2.1.14 Lemma *The mapping $i_* : Z_*\left(\coprod_\nu X_\nu\right) \longrightarrow Z_*(X)$ given by*

$$i_*\Big(\sum_{k\in I} n_k Z_k\Big) = \sum_{k\in I} n_k\, i(Z_k)$$

is continuous and finite.

For the proof see [Sie94, Lemma 1.6].

2.1.15 Definition If $\mathcal{S} = (\chi, W, D)$ is a d-dimensional scale, φ a continuous (complex-valued) function on $|\mathcal{S}|$ and $t \in D$, then the mapping $\varphi_t^\sharp : B_\mathcal{S} \longrightarrow \mathbb{C}$ is defined by

$$\varphi_t^\sharp\Big(\sum_{k\in I} n_k Z_k\Big) := \sum_{\{k|\dim Z_k = d\}} n_k \cdot \left(\sum_{x\in Z_k\cap V_t} o(Z_k, x)\cdot\varphi(x)\right),$$

where $V_t := \chi^{-1}(W \times \{t\})$ and $o(Z_k, x)$ is the ramification order[4] in x of the analytic covering $\mathrm{pr}_2 \circ \chi : Z_k \cap |\mathcal{S}| \longrightarrow D$.

Note that taking $\varphi \equiv 1$,

$$1_t^\sharp(Z) = \deg_\mathcal{S}(Z) \quad \forall t \in D,\ Z \in B_\mathcal{S}.$$

Siebert proves in [Sie94, Lemma 1.8]) the following

2.1.16 Lemma *$\varphi_t^\sharp : B_\mathcal{S} \longrightarrow \mathbb{C}$ is continuous.*

The nice thing about these functions is that they separate the cycles locally, even with φ holomorphic:

2.1.17 Lemma *If $\mathcal{S} = (\chi, W, D)$ is a d-dimensional scale and if $Z_1, Z_2 \in B_\mathcal{S}$ are two cycles such that $Z_1^{(d)}|_{|\mathcal{S}|} \neq Z_2^{(d)}|_{|\mathcal{S}|}$, then there exist a holomorphic function $\varphi \in \mathcal{O}(|\mathcal{S}|)$ and $t \in D$ such that $\varphi_t^\sharp(Z_1) \neq \varphi_t^\sharp(Z_2)$.*

Proof Without loss of generality we assume that $Z_k = Z_k^{(d)}$. If $|Z_1||_{|\mathcal{S}|} \neq |Z_2||_{|\mathcal{S}|}$, then there exists $t \in D$ with $|Z_1| \cap V_t \neq |Z_2| \cap V_t$ ($V_t := \chi^{-1}(W \times \{t\})$). Since $\mathrm{pr}_2 \circ \chi : |Z_k| \cap |\mathcal{S}| \longrightarrow D$ are coverings, the set $(|Z_1| \cup |Z_2|) \cap V_t$ is finite. Since $|\mathcal{S}|$ is a Stein Space, there exists a holomorphic function $\varphi \in \mathcal{O}(|\mathcal{S}|)$ such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in V_t \cap (|Z_1| \smallsetminus |Z_2|) \\ 0 & \text{if } x \in V_t \cap |Z_2|. \end{cases}$$

Thus $\varphi_t^\sharp(Z_1) > 0$ and $\varphi_t^\sharp(Z_2) = 0$.
If $|Z_1||_{|\mathcal{S}|} = |Z_2||_{|\mathcal{S}|}$, then $Z_1|_{|\mathcal{S}|}$ and $Z_2|_{|\mathcal{S}|}$ have a common irreducible component S such that the coefficient μ_1 of S in the cycle $Z_1|_{|\mathcal{S}|}$ is different from the coefficient μ_2 of S in the cycle $Z_2|_{|\mathcal{S}|}$. We choose $t \in D$ such that the analytic covering $\eta = \mathrm{pr}_2 \circ \chi : |Z_1| \cap |\mathcal{S}| \longrightarrow D$ is not ramified in each point of $\eta^{-1}(t)$. We write $\eta^{-1}(t) = \{x_1, \ldots, x_b\}$ with $x_1 \in S$. There exists an holomorphic function $\varphi \in \mathcal{O}(|\mathcal{S}|)$ such that $\varphi(x_1) = 1$ and $\varphi(x_k) = 0$ for $k > 1$. Thus

$$\varphi_t^\sharp(Z_1) = \mu_1 \neq \varphi_t^\sharp(Z_2) = \mu_2,$$

which concludes the proof. □

[4] If $\eta : X \longrightarrow Y$ is an analytic covering with Y irreducible, then the ramification order of η in a point $x \in X$ is k if there exists a fundamental system of open neighbourhoods of x in X such that $\eta|_U$ is a k-sheeted analytic covering for each U in the fundamental system. Note that if η is not ramified in x, then the ramification order of η in x is 1.

2.2 Analytic families of cycles and the Barlet space $\mathcal{B}_d(X)$

This subsection describes the notion of analytic family of cycles and the way to impose a complex structure on the set of d-dimensional compact analytic cycles to obtain the d-dimensional Barlet space $\mathcal{B}_d(X)$. This theory was introduced by Barlet in [Bar75].

In this subsection, X denotes a complex space.

Let Y be a complex space and $\rho : Y \longrightarrow Z_*(X)$ be a continuous mapping. Choose a d-dimensional scale $\mathcal{S} := (\chi : V \longrightarrow \Omega, W, D)$ and a natural number k.

Denote $U := \rho^{-1}(B_{\mathcal{S}}(k)) \Subset Y$. The following describes the construction of the mapping

$$\sum\nolimits_{\mathcal{S}} : U \times D \longrightarrow \mathrm{Sym}^k(W)$$

where $\mathrm{Sym}^k(W)$ is the quotient of $W^k = W \times \cdots \times W$ by the group S_k of permutations. By a theorem of H. Cartan on the quotient of complex manifolds, $\mathrm{Sym}^k(W)$ is a normal complex space (see [KK83, Corollary 72.5]).

Let $(y, t) \in U \times D$. We denote $\rho(y)^{(d)} =: \sum_i n_i Z_i$. The cycle $\chi_*(\rho(y)^{(d)})$ is a d-dimensional cycle on Ω. Let

$$A := \left|\chi_*(\rho(y)^{(d)})\right| \cap (W \times \{t\}) =: \{(w_1, t), \dots (w_l, t)\}.$$

For each $j = 1, \dots, l$, we define a number m_j in the following way: let $W_j \Subset W$ be an open neighbourhood of w_j and $D_j \Subset D$ be an open neighbourhood of t such that

– $W_j \cap \mathrm{pr}_1(A) = \{w_j\}$ (where $\mathrm{pr}_1 : W \times D \longrightarrow W$ is the projection on the first factor)
– $\mathcal{S}_j := (\chi, W_j, D_j)$ is a scale adapted to $\rho(y)^{(d)}$
– $W_{j_1} \cap W_{j_2} = \emptyset$ if $j_1 \neq j_2$.

Then we set $m_j := \deg_{\mathcal{S}_j}(\rho(y)^{(d)})$.

2.2.1 Claim $\sum_{j=1}^{l} m_j = k$

Proof By definition, $\sum m_j = \sum_j \sum_i n_i \nu_{i,j}$ where $\nu_{i,j}$ is the number of sheets of the covering $Z_i \cap |\mathcal{S}_j| \to D_j$. Let $t_0 \in \bigcap D_j$ be an unbranched point of $Z_i \cap |\mathcal{S}| \to D$ and of $Z_i \cap |\mathcal{S}_j| \to D_j$ for all i, j. Then

$$\begin{aligned} \sum_j \sum_i n_i \nu_{i,j} &= \sum_i n_i \sum_j \mathrm{Card}\left((W_j \times \{t_0\}) \cap \chi(Z_i)\right) \\ &= \sum_i n_i \,\mathrm{Card}\left((W \times \{t_0\}) \cap \chi(Z_i)\right) = k \end{aligned}$$

which concludes the proof. □

Then $\sum_{\mathcal{S}}$ is defined by

$$\sum\nolimits_{\mathcal{S}}(y,t) := [(\underbrace{w_1,\dots,w_1}_{m_1 \text{ times}},\dots,\underbrace{w_l,\dots,w_l}_{m_l \text{ times}})] \in \mathrm{Sym}^k(W).$$

2.2.2 Definition *An **analytic family** of cycles on X is a continuous mapping $\rho\colon Y \longrightarrow Z_*(X)$, where Y is a complex space, such that for each scale $\mathcal{S}$ and for each $k \in \mathbb{N}$, the mapping $\sum_{\mathcal{S}}$ is holomorphic.*

2.2.3 Lemma *If $\rho\colon Y \longrightarrow Z_d(X)$ is an analytic family, then $\rho'\colon Y \longrightarrow Z_d(X)$ given by $\rho'(y) := M \cdot \rho(y)$ is an analytic family for each $M \in \mathbb{N}_{>0}$*

Proof For a scale $\mathcal{S}$ and $k \in \mathbb{N}$,

$$B_{\mathcal{S}}(k) \cap \rho'(Y) = \begin{cases} B_{\mathcal{S}}(l) \cap \rho(Y) & \text{if } k = l \cdot M \\ \emptyset & \text{else} \end{cases}$$

and thus

$$(\rho')^{-1}(B_{\mathcal{S}}(k)) = \begin{cases} \rho^{-1}(B_{\mathcal{S}}(l)) & \text{if } k = l \cdot M \\ \emptyset & \text{else.} \end{cases}$$

Hence $(\rho')^{-1}(B_{\mathcal{S}}(k))$ is open in X by the continuity of ρ. Thus ρ' is continuous. Let $y_0 \in Y$, let $\mathcal{S}$ be a scale adapted to $\rho(y_0)$ and let $k := \deg_{\mathcal{S}}((\rho(y_0))$. Since ρ' is continuous, $(\rho')^{-1}(B_{\mathcal{S}}(Mk))$ is an open neighbourhood of y_0 in Y. One sees that the mapping

$$\sum\nolimits_{\mathcal{S}}'\colon U \times D \longrightarrow \mathrm{Sym}^{Mk}(W)$$

is given by $\sum'_{\mathcal{S}} = \varphi \circ \sum_{\mathcal{S}}$ where $\varphi\colon \mathrm{Sym}^k(W) \longrightarrow \mathrm{Sym}^{Mk}(W)$ is given as follows:

$$\varphi[(w_1,\dots,w_k)] = [(\underbrace{w_1,\dots,w_1}_{M \text{ times}},\dots,\underbrace{w_k,\dots,w_k}_{M \text{ times}})].$$

But, φ is holomorphic, which proves the holomorphy of $\sum'_{\mathcal{S}}$. □

2.2.4 Definition A family $\rho\colon Y \longrightarrow Z_d(X)$ of cycles is called ***proper*** if for each $y_0 \in Y$, there exist a relatively compact open neighbourhood $U \Subset X$ of $|\rho(y_0)|$ and an open neighbourhood $V \Subset Y$ of y_0 such that if $y \in V$ then $|\rho(y)| \subset U$.

Note that if a family is proper then each analytic cycle in the family is compact (a cycle is called ***compact*** if its support is a compact analytic set).

2.2.5 Remark Our terminology is different from the terminology of Barlet. We adopt the terminology used by Siebert in [Sie94]. Our notion of "analytic family of cycles" corresponds to Barlet's notion of "*local* analytic family of cycles"; Barlet's notion of "analytic family of cycles" corresponds to our notion of "*proper* analytic family of cycles".

The set
$$C_d(X) := \{Z \in Z_d(X) \mid |Z| \text{ is compact}\}.$$
is the set of d-dimensional compact analytic cycles.

In [Bar75], Barlet proves that $C_d(X)$ admits a complex structure:

2.2.6 Theorem (Barlet) *There exists exactly one complex space $\mathcal{B}_d(X)$ such that its underlying set is $C_d(X)$ and such that for each complex space Y and each mapping $\rho : Y \longrightarrow C_d(X)$, the following conditions are equivalent:*

- *ρ is a proper analytic family of cycles*
- *$\rho : Y \longrightarrow \mathcal{B}_d(X)$ is a holomorphic mapping.*

We call $\mathcal{B}_d(X)$ the d-dimensional ***Barlet space*** on X.

2.2.7 Remark In [Bar75, Page 102], Barlet remarks that the canonical inclusion of $\mathcal{B}_d(X)$ in $Z_d(X)$ is continuous, but not open on its image.

2.2.8 Convention In the following, $C_d(X)$ denotes the topological space of compact analytic cycles with the topology of Barlet, i.e. the topology induced from $Z_d(X)$, which is defined in subsection 2.1. By the previous remark, the topology of $C_d(X)$ is different from the topology of $\mathcal{B}_d(X)$. The identity mapping $\mathcal{B}_d(X) \to C_d(X)$ is continuous, but not a homeomorphism.

2.3 Multiplicities of fibres and generic locus of mappings

In the first part of this subsection the notion of multiplicity of fibres of an open mapping is presented. This notion was first introduced by Stoll and next by Tung ([Sto66], [Tun79]). The current description is based on [Sie94, Section 2]. In a second part, the link between this multiplicity and the Barlet-topology is explained.

2.3.1 Lemma *If $f : X \longrightarrow Y$ is an open mapping from a locally pure dimensional complex space X to a locally irreducible complex space Y, then the dimension of the fibres of f is locally constant, i.e. for each $x \in X$ there exists an open neighbourhood $V \Subset X$ of x such that* $\dim (f^{-1}(f(x)) \cap V) = \dim V - \dim f(V)$.

Proof Let $x \in X$ and $y := f(x)$. There exists a connected open neighbourhood $V \Subset X$ of x such that $\dim_v X = \dim V$ for each $v \in V$ and $f(V) \Subset Y$ is irreducible. Since f is open and Y is locally irreducible, the following equation holds for each $a \in X$ (see [Fis76, 3.10]):
$$\dim_a X = \dim_{f(a)} Y + \dim_a f^{-1}(f(a)).$$
In our case, for each $v \in f^{-1}(y) \cap V$,
$$\dim V = \dim_v X = \dim_y Y + \dim_v f^{-1}(y).$$

Thus $\dim_v f^{-1}(y) = \dim V - \dim_y Y$ and hence

$$\dim(f^{-1}(y) \cap V) = \dim V - \dim f(V),$$

which concludes the proof. □

In the following construction, f denotes an open mapping from a locally pure dimensional complex space X to a locally irreducible complex space Y. Let $S \subset X$ be an irreducible component of a fibre $f^{-1}(y)$ over some $y \in Y$. The following describes a procedure to assign a positive integer to S, called its ***multiplicity*** $\mu_f(S)$:

Let $x \in S$ be a point in the regular locus of $f^{-1}(y)$. Then there exist an open neighbourhood $V \Subset X$ of x and an embedding $i: V \longrightarrow \mathbb{C}^N$ ($N := \operatorname{emdim}_x X$) such that $V \cap f^{-1}(y)$ is a subset of the regular locus of $f^{-1}(y)$. We can suppose that $V \cap f^{-1}(y) = V \cap S$. There exists a projection $p: \mathbb{C}^N \longrightarrow \mathbb{C}^d$ ($d := \dim S$) such that $\rho|_{S\cap V}: S \cap V \longrightarrow \mathbb{C}^d$ is biholomorphic onto its image, where $\rho: V \longrightarrow \mathbb{C}^d$ is given by $\rho := p \circ i$ (we restrict V if necessary). Consider the mapping

$$\begin{array}{rcccl} (f,\rho) & : & V & \to & Y \times \mathbb{C}^d \\ & & v & \mapsto & (f(v), \rho(v)). \end{array}$$

The fibre of (f, ρ) over $(y, \rho(x))$ is given by

$$\begin{aligned} (f,\rho)^{-1}(y,\rho(x)) &= f^{-1}(y) \cap \rho^{-1}(\rho(x)) \\ &= (f^{-1}(y) \cap V) \cap \{v \in V \mid \rho(v) = \rho(x)\} \\ &= \{v \in S \cap V \mid \rho(v) = \rho(x)\} = \{x\}. \end{aligned}$$

Thus, by [GR84, 3.1.2], there exist an open irreducible neighbourhood $U \Subset Y$ of y and an open neighbourhood $D \Subset \mathbb{C}^d$ of $\rho(x)$ such that the mapping $(f, \rho): V \longrightarrow U \times D$ is finite, where $V = (f, \rho)^{-1}(U \times D)$. Since $\dim V = \dim U + \dim D$ and since U is irreducible, (f, ρ) is a sheeted analytic covering. Denote

$$\mu_f(S) := \text{number of sheets of } (f, \rho).$$

The fact that $\mu_f(S)$ does not depend of the choice of ρ and y was proved by Tung (see [Tun79, Theorem 2.2.1 and Proposition 2.2.6]).

2.3.2 Example Let $f: \mathbb{C}^2 \longrightarrow \mathbb{C}$ be given by $f(z) := z_1^2 z_2^3$. Denoting $A := \{z \in \mathbb{C}^2 \mid z_2 = 0\}$ and $B := \{z \in \mathbb{C}^2 \mid z_1 = 0\}$, $f^{-1}(0) = A \cup B$. Let $x := (2, 0) \in A$. Let $D \Subset \mathbb{C}$ be the unit disk centered at 0, $D' \Subset \mathbb{C}$ be the unit disk centered at 2 and $V := f^{-1}(D) \cap (D' \times \mathbb{C})$. If $\rho: V \longrightarrow D'$ is given by $\rho(z) := z_1$, then $(f, \rho)(z) = (z_1^2 z_2^3, z_1)$ is a 3-sheeted analytic covering. Thus $\mu_f(A) = 3$. Similarly, one calculates that $\mu_f(B) = 2$.

2.3.3 Definition Let $f: X \longrightarrow Y$ be a generically open holomorphic mapping from a complex space X to a complex space Y. For each irreducible component X_ν of X, let Y_ν be the irreducible component that contains $f(X_\nu)$ (see lemma 1.2.2). Define

$$E_\nu := \{x \in X_\nu \mid \dim_x f^{-1}(f(x)) > \dim X_\nu - \dim Y_\nu\},$$

$E := \bigcup E_\nu$ and $\widetilde{E} := f^{-1}(f(E) \cup N)$, where N is the non-normal locus of Y. Then the set

$$Y_{\mathrm{gen}(f)} := f(X \smallsetminus \widetilde{E}) = f(X) \smallsetminus (f(E) \cup N)$$

is called the ***generic locus*** of f.

In general $Y_{\mathrm{gen}(f)}$ is not open in Y or in $\operatorname{Im} f$, but it is a dense subset of $\operatorname{Im} f$ (the last assertion is a consequence of [Sie93, Lemma 1.1]).

2.3.4 Remark If Y is a normal complex space and $f\colon X \longrightarrow Y$ is open, then $E = \emptyset$ and $N = \emptyset$. Thus $Y_{\mathrm{gen}(f)} = f(X)$.

By proposition 1.2.3, any fibre $f^{-1}(y)$ over $y \in Y_{\mathrm{gen}(f)}$ has an open neighbourhood $V \subset X$ disjoint from $E \cup f^{-1}(N)$ such that $f|_V\colon V \longrightarrow Y \smallsetminus N$ is an open mapping. If X is locally pure dimensional, then the above procedure lets us assign multiplicities on irreducible components of the fibres of f, that defines a mapping

$$Z_{f,\mathrm{gen}}\colon Y_{\mathrm{gen}(f)} \longrightarrow Z_*(X), \quad x \mapsto \sum_{S \subset f^{-1}(y)} \mu_f(S)[S].$$

If X is not locally pure dimensional, X is decomposed into irreducible components X_ν and the mapping $f \circ i\colon \coprod_\nu X_\nu \longrightarrow Y$, where $i\colon \coprod_\nu X_\nu \longrightarrow X$ is the mapping given by the canonical inclusions $i_\nu\colon X_\nu \longrightarrow X$. In this case, Siebert defines

$$Z_{f,\mathrm{gen}} := i_* \circ Z_{f\circ i,\mathrm{gen}}\colon Y_{\mathrm{gen}(f)} \longrightarrow Z_*(X).$$

(See lemma 2.1.14 for the definition of i_*.)

2.3.5 Example Let $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ be given by $f(z) := z_1^2 z_2^3$. Since f is open and surjective, $\mathbb{C}_{\mathrm{gen}(f)} = \mathbb{C}$ by remark 2.3.4. Doing the same calculation of multiplicities than in example 2.3.2, the mapping $Z_{f,\mathrm{gen}}\colon \mathbb{C} \longrightarrow Z_1(\mathbb{C}^2)$ is given by

$$Z_{f,\mathrm{gen}}(c) = \begin{cases} [f^{-1}(c)] & \text{if } c \neq 0 \\ 3[A] + 2[B] & \text{if } c = 0, \end{cases}$$

where $A := \{z_2 = 0\}$ and $B := \{z_1 = 0\}$.

2.3.6 Remark By construction, if $y \in Y_{\mathrm{gen}(f)}$, then $|Z_{f,\mathrm{gen}}(y)| = f^{-1}(y)$.

The following Lemma shows the relation between the fibre multiplicities and the Barlet-topology

2.3.7 Lemma *The mapping $Z_{f,\mathrm{gen}}\colon Y_{\mathrm{gen}(f)} \longrightarrow Z_*(X)$ is a continuous injection.*

For the proof see [Sie94, Lemma 3.1].

The following lemma is helpful for the following:

2.3.8 Lemma *Let $f\colon X \longrightarrow Y$ and $\sigma\colon Y' \longrightarrow Y$ be generically open holomorphic mappings, let $X' \subset Y' \times_Y X$ be the union of irreducible components projecting generically open on Y', let $f'\colon X' \longrightarrow Y'$ be the canonical projection and let N' bet the non-normal locus of Y'. Then $\sigma^{-1}(Y_{\mathrm{gen}(f)}) \cap (Y' \smallsetminus N')$ is contained in $Y'_{\mathrm{gen}(f')}$ and the following holds:*

$$Z_{f',\mathrm{gen}}(y') = [y'] \times Z_{f,\mathrm{gen}}(\sigma(y)) \quad \forall y' \in \sigma^{-1}(Y_{\mathrm{gen}(f)}) \cap (Y' \smallsetminus N').$$

For the proof see [Sie94, Lemma 3.3].

2.4 The fibre-cycle space $Z(f)$

This subsection explains the notion of the fibre-cycle space $Z(f)$ of a generically open holomorphic mapping f. At the end, we present a condition introduced by Siebert with which this topological space is a complex space.

2.4.1 Definition If $f\colon X \longrightarrow Y$ is a generically open holomorphic mapping. Then the ***fibre-cycle space*** $Z(f)$ of f is defined by

$$Z(f) := \overline{\mathrm{Im}(Z_{f,\mathrm{gen}})} \smallsetminus \{[\emptyset]\} \subset Z_*(X)$$

with the induced topology. An element $Z \in Z(f)$ is called a ***fibre-cycle***. The fiber-cycle Z is called ***generic*** if $Z \in \mathrm{Im}(Z_{f,\mathrm{gen}})$.

2.4.2 Remark If $\mathrm{Im}(f)$ is not compact, then $[\emptyset] \in \mathrm{Im}(Z_{f,\mathrm{gen}})$ (compare example 2.1.8). That's why we drop $[\emptyset]$ in the definition of Z_f.

2.4.3 Lemma *If $Z \in Z(f)$, then $f(x) = f(x')$ for each $x, x' \in |Z|$.*

Proof By construction, if $Z = Z_{f,\mathrm{gen}}(y) \in \mathrm{Im}(Z_{f,\mathrm{gen}})$, then $|Z| = f^{-1}(y)$ and $f(x) = y$ for each $x \in |Z|$. If $Z \in Z(f)$, then there exists a sequence (Z_k) of cycles in $\mathrm{Im}(Z_{f,\mathrm{gen}})$ such that $Z_k \to Z$. By proposition 2.1.11, $|Z_k| \to |Z|$. Let $x \in |Z|$. Then there exists a sequence (x_k) with $x_k \in Z_k$ for all k and $x_k \to x$. Let $x' \in |Z|$. There exists a sequence (x'_k) with $x'_k \in |Z_k|$ and $x'_{k_l} \to x'$ for a subsequence (x'_{k_l}). Hence

$$f(x') = f(\lim x'_{k_l}) = f(\lim x_{k_l}) = f(x)$$

which concludes the proof. $\square$

2.4.4 Lemma *If $Z \in Z(f)$, then $y := f(|Z|) \in Y$ is called the* ***base point*** *of Z. This construction defines a continuous mapping*

$$\sigma\colon Z(f) \longrightarrow Y.$$

The restriction $\sigma\colon \mathrm{Im}(Z_{f,\mathrm{gen}}) \longrightarrow Y_{\mathrm{gen}(f)}$ is the inverse mapping of $Z_{f,\mathrm{gen}}$.

Proof The mapping is well-defined by lemma 2.4.3. The continuity is a consequence of lemma 2.1.12 since the equation

$$\sigma^{-1}(U) = \{Z \in Z_*(X) \mid |Z| \cap f^{-1}(U) \neq \emptyset\} \cap Z(f)$$

holds. □

2.4.5 Lemma *If $K \subset X$ is compact, then*

$$\{Z \in Z(f) \mid |Z| \cap K \neq \emptyset\} \subset Z(f)$$

is compact in $Z(f)$.

For the proof see [Sie94, Remark 7.5]

2.4.6 Proposition $X = \bigcup\limits_{Z \in Z(f)} |Z|$.

Proof If $x \in X$, then there exists a sequence (x_k) in $X \smallsetminus \widetilde{E}$ such that $x_k \to x$, since $X \smallsetminus \widetilde{E}$ is dense in X by [Sie93, Lemma 1.1] (See definition 2.3.3 for the definition of $\widetilde{E}$). By lemma 2.4.5, the corresponding sequence $(Z_{f,\mathrm{gen}}(f(x_k)))$ of fibre-cycles is relatively compact in $Z(f)$. Thus, there exists a subsequence that converges to a fibre-cycle Z, and hence $x \in |Z|$. □

The structure sheaf ${}_{Z(f)}\mathcal{O}$ on $Z(f)$ is given by the sheaf associated to the presheaf $({}_{Z(f)}\mathcal{O}(U))_{U \Subset Z(f)}$ where

$$\varphi : U \longrightarrow \mathbb{C} \in {}_{Z(f)}\mathcal{O}(U) \quad :\Longleftrightarrow \quad \begin{cases} \text{There exists } \psi : V := (\sigma(U))^\circ \longrightarrow \mathbb{C} \in \\ {}_Y\widehat{\mathcal{O}}(V) \text{ such that } \varphi|_{\sigma^{-1}(V)} = \psi \circ \sigma. \end{cases}$$

With that construction, $(Z(f), {}_{Z(f)}\mathcal{O})$ is a ringed space.

We are now able to present the main result of [Sie94], saying when $Z(f)$ with the above structure is a complex space.

2.4.7 Definition A generically open holomorphic mapping $f : X \longrightarrow Y$ is called ***fibre-cycle separable*** if for each $Z \in Z(f)$, there exist an open neighbourhood $U \Subset Z(f)$ of Z and a relatively compact open subset $B \Subset X$ of X such that if $Z_1, Z_2 \in U$ with $Z_1|_B = Z_2|_B$, then $Z_1 = Z_2$.

2.4.8 Theorem *If f is a generically open holomorphic mapping that is fibre-cycle separable, then the ringed space $(Z(f), {}_{Z(f)}\mathcal{O})$ is a complex space.*

The proof is complicated. It is given by Siebert in [Sie94, 7.3]. The main part of the proof is the use of the theorem "Lemma (n)" (see [Sie92, 5.1] or [Sie93, 2.1]) proven for the first time by Grauert.

2.5 Geometrically flat mappings and geometrically flat equivalence relations

The first part of this subsection explains the notion of geometrically flat mappings and connects this notion with the analytic families. The second part presents what is a geometrically flat equivalence relation and states that if X is maximal then the quotient of X by a geometrically flat equivalence relation is a complex space (compare theorem 2.5.9).

2.5.1 Definition *A generically open holomorphic mapping $f\colon X \longrightarrow Y$ from a complex space X to a maximal complex space Y is called **geometrically flat** if the continuous mapping*

$$Z_{f,\mathrm{gen}}\colon Y_{\mathrm{gen}(f)} \longrightarrow Z_*(X) \smallsetminus \{[\emptyset]\}$$

has a continuous extension

$$Z_f\colon Y \longrightarrow Z_*(X) \smallsetminus \{[\emptyset]\}.$$

2.5.2 Remark If f is geometrically flat, then f is surjective, using the fact that $[\emptyset] \notin \mathrm{Im}(Z_f)$. Furthermore, for each $y \in Y$, we prove that the equation $|Z_f(y)| = f^{-1}(y)$ holds. It holds on $Y_{\mathrm{gen}(f)}$. Let $y \in Y$ and $x \in f^{-1}(y)$. Since $f^{-1}(Y_{\mathrm{gen}(f)})$ is dense in X (see the construction of $Y_{\mathrm{gen}(f)}$), there exists a sequence (x_n) in $f^{-1}(Y_{\mathrm{gen}(f)})$ such that $x_n \to x$. Hence $y_n := f(x_n) \to y$. Thus

$$x = \lim x_n \in \lim |Z_{f,\mathrm{gen}}(y_n)| = |Z_f(y)|$$

showing that $f^{-1}(y) \subset |Z_f(y)|$. Conversely,

$$|Z_f(y)| = \lim |Z_{f,\mathrm{gen}}(y_n)| = \lim f^{-1}(y_n) \subset f^{-1}(y).$$

2.5.3 Proposition *Geometrically flat mappings are open.*

Proof Let $f\colon X \longrightarrow Y$ be geometrically flat. Let $U \Subset X$. By lemma 2.1.12, the set $V := \{Z \in Z_*(X) \mid |Z| \cap U \neq \emptyset\}$ is open and thus $Z_f^{-1}(V)$ is open. We have to show that $Z_f^{-1}(V) = f(U)$. But this equation holds by remark 2.5.2. □

In general, an open surjective mapping is not geometrically flat, but

2.5.4 Lemma *If $f\colon X \longrightarrow Y$ is an open surjection and Y is normal, then f is geometrically flat.*

Proof Since f is open and Y is normal, then $Y_{\mathrm{gen}(f)} = f(X) = Y$ by remark 2.3.4. So $Z_{f,\mathrm{gen}} = Z_f$ is continuous. □

The next two propositions give a relationship between analytic families of cycles and geometrically flat mappings:

2.5.5 Proposition *Let $\rho : S \longrightarrow Z_*(X)$ be a continuous family of generically reduced analytic cycles (i.e. the cycles $\rho(s)$ are reduced for s outside some thin analytic subset of S) on X such that S is a maximal complex space. Then the following conditions are equivalent:*

(a) *ρ is an analytic family*

(b) *There exists a closed subspace $L \subset S \times X$ such that the projection $p_1 : L \longrightarrow S$ is geometrically flat and such that $\rho(s) = p_{2*}(Z_{p_1}(s))$ for each $s \in S$, where $p_2 : L \longrightarrow X$ is the projection on X*

(c) *For each scale $\mathcal{S}$ and each $\varphi \in \mathcal{O}(|\mathcal{S}|)$, the function $\varphi_t^\sharp \circ \rho : \rho^{-1}(B_{\mathcal{S}}) \longrightarrow \mathbb{C}$ is a holomorphic function.*

For the proof see [Sie94, Proposition 5.3]. □

2.5.6 Corollary *If $f : X \longrightarrow Y$ is geometrically flat, then $Z_f : Y \longrightarrow Z_*(X)$ is an analytic family of cycles on X.*

Proof Let $L \subset Y \times X$ be the graph of f. Hence the condition (b) of proposition 2.5.5 is satisfied. □

An important application of geometrically flat mappings is the notion of geometrically flat equivalence relations:

2.5.7 Definition An analytic equivalence relation R on a maximal complex space X is called ***geometrically flat*** if the projections $p_j : R \longrightarrow X$, $j = 1, 2$, are geometrically flat.

2.5.8 Lemma *If R is a geometrically flat analytic equivalence relation, then the equation $|p_{1*}(Z_{p_2}(x))| = R(x)$ holds for each $x \in X$.*

For the proof see [Sie92, Lemma 7.2]. □

2.5.9 Theorem *If R is a geometrically flat analytic equivalence relation on a maximal space X, then X/R is a complex space*

For the proof see [Sie92, Theorem 7.3].

2.5.10 Corollary *If R is a geometrically flat equivalence relation, then the canonical projection $X \to X/R$ is geometrically flat. Conversely, if $f : X \longrightarrow Y$ is geometrically flat and X is maximal, then the equivalence relation R_f on X is geometrically flat.*

For the proof see [Sie92, Corollary 7.5]

2.5.11 Remark Geometrically flatness is a sufficient but not necessary condition for an open analytic equivalence relation to have a complex structure on X/R, as it is shown by example 2.5.12 due to Bohnhorst-Reiffen (see [BR90]).

2.5.12 Example Let $A = X^{(1)} = X^{(2)} := \{x \in \mathbb{C}^2 \mid x_1 x_2 = 0\}$ and $X := X^{(1)} \amalg X^{(2)}$. The elements of $X^{(j)}$ are denoted by $(x_1, x_2)^{(j)}$. Denote $a := (0,0)^{(1)}$ and $b := (0,0)^{(2)}$. Consider the equivalence relation R on X given by:

$$(t,0)^{(1)} \sim (-t,0)^{(1)} \sim (t^2,0)^{(2)} \quad \text{and} \quad (0,t^2)^{(1)} \sim (0,t)^{(2)} \sim (0,-t)^{(2)}.$$

Figure 2 shows that equivalence relation.

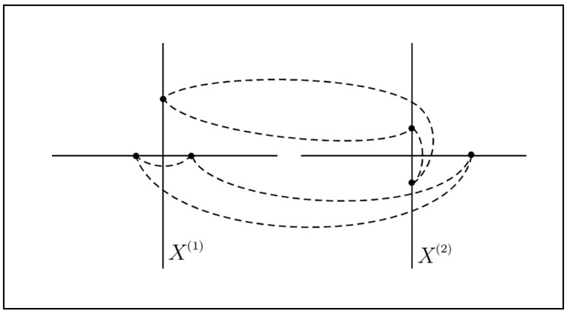

Figure 2: The equivalence relation of example 2.5.12

One sees that R is finite, open and analytic. Let $\varphi: X \longrightarrow A$ be the mapping given by $\varphi(x^{(1)}) = (x_1^2, x_2)$ and $\varphi(x^{(2)}) = (x_1, x_2^2)$. This mapping is surjective and $R = R_\varphi$. Thus φ induces an isomorphism of ringed space $\overline{\varphi}: X/R \longrightarrow A$. Hence X/R is a complex space.

The mapping φ is not geometrically flat: One sees that $A_{\mathrm{gen}(\varphi)} = A \setminus (0,0)$. For $t \neq 0$,

$$\begin{aligned} Z_{\varphi,\mathrm{gen}}(t,0) &= [(\sqrt{t},0)^{(1)}] + [(-\sqrt{t},0)^{(1)}] + [(t,0)^{(2)}] \quad \text{and} \\ Z_{\varphi,\mathrm{gen}}(0,t) &= [(0,\sqrt{t})^{(2)}] + [(0,-\sqrt{t})^{(2)}] + [(0,t)^{(1)}], \end{aligned}$$

and thus

$$2[a] + [b] = \lim Z_{\varphi,\mathrm{gen}}(1/n,0) \neq \lim Z_{\varphi,\mathrm{gen}}(0,1/n) = [a] + 2[b].$$

Hence, since φ is not geometrically flat, R is not geometrically flat by corollary 2.5.10.

2.6 Meromorphic equivalence relations

This subsection presents the notion of meromorphic equivalence relations, an important application of the theory of analytic cycles. This notion was introduced by Grauert in [Gra86]. The main reference for this subsection is [Sie92, §8].

In this subsection, X is a complex space

2.6.1 Definition An analytic subset $R \subset X \times X$ is called a ***meromorphic equivalence relation on X*** if there exists a thin analytic subset $P \subset X$ (the ***polar set***) such that

(a) $X \setminus P$ is normal

(b) $R \cap ((X \setminus P) \times (X \setminus P))$ is dense in R

(c) $R|_{X \setminus P}$ is a geometrically flat equivalence relation on $X \setminus P$.

2.6.2 Remark The condition (c) of the previous definition is equivalent to the fact that $R|_{X \setminus P}$ is open, since $X \setminus P$ is normal.

The following lemma explains how to associate to each generically open holomorphic mapping a meromorphic equivalence relation, which is used in [Sie92, Satz 8.6]. In general, the equivalence relation of a generically open mapping is not a meromorphic equivalence relation, as it is shown by example 2.6.4.

2.6.3 Lemma *If $f : X \longrightarrow Y$ is a generically open holomorphic mapping, then the union of those irreducible components of R_f, for which the two projections on X are generically open, is a meromorphic equivalence relation.*

Proof Let $\widetilde{R}$ be the union of those irreducible components of R_f, for which the two projections on X are generically open. Since f is generically open, there exists a thin analytic subset P of X such that $X \setminus P$ is normal and $f' := f|_{X \setminus P}$ is open. We prove that $\widetilde{R}|_{X \setminus P} = R_{f'}$ and $\overline{R_{f'}} = \widetilde{R}$, which implies that $\widetilde{R}$ is a meromorphic equivalence relation.
Let R_ν be an irreducible component of $R_{f'} = R_f|_{X \setminus P} \supset \widetilde{R}|_{X \setminus P}$. Then the two projections $p_j : R_\nu \longrightarrow X \setminus P$ are generically open. Hence $R_\nu \subset \widetilde{R}$. Thus $R_{f'} = \widetilde{R}|_{X \setminus P}$.
Suppose that $\widetilde{R} \not\subset \overline{\widetilde{R}|_{X \setminus P}} = \overline{R_{f'}}$. Since $\overline{R_{f'}}$ is closed in $X \times X$ and $\overline{R_{f'}} \subset \widetilde{R}$, we obtain that $\overline{R_{f'}}$ is closed in $\widetilde{R}$. Let U be the open subset of $\widetilde{R}$ given by $U := \widetilde{R} \setminus \overline{R_{f'}}$. Let V be a connected component of $U \setminus \operatorname{Sing} \widetilde{R}$. Then $V \subset X \times P$ or $V \subset P \times X$. We can suppose that $V \subset P \times X$. Then $p_1(V) \subset P$, and thus $p_1(V)$ is nowhere dense in X. Let R_ν be the irreducible component of $\widetilde{R}$ that contains V. Then $p_1|_{R\nu}$ is not generically open, which is a contradiction with the definition of $\widetilde{R}$. Hence $\widetilde{R} = \overline{R_{f'}}$, which completes the proof. □

2.6.4 Example Let $f : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ be given by $f(z_1, z_2, z_3) = (z_1 z_3, z_2 z_3)$. The fibres $f^{-1}(x, y)$ over $(x, y) \neq (0, 0)$ have pure dimension 1. The fiber over $(0, 0)$ is given by

$$f^{-1}(0,0) = \{z \in \mathbb{C}^3 \mid z_3 = 0\} \cup \{z \in \mathbb{C}^3 \mid z_1 = z_2 = 0\}.$$

Hence $\dim_{(z_1,z_2,0)} f^{-1}(0,0) = 2$. Hence, by [Fis76, Theorem 3.10], f is not open, but only generically open.
The equivalence relation R_f associated to f is not open. A simple calculation shows that

$$R_f \quad = \quad \{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid z_1 z_3 = w_1 w_3,\ z_2 z_3 = w_2 w_3\}$$

$$= \overbrace{\{(z,w) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid z_1z_3 = w_1w_3,\ z_2z_3 = w_2w_3,\ z_2w_1 = z_1w_2\}}^{:=R'} \\ \cup \underbrace{\{(z,w) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid z_3 = w_3 = 0\}}_{:=A}$$

Since $p_1(A) = \mathbb{C}^2 \times \{0\}$, $p_1|_A$ is not generically open. Denote $P := \{z \in \mathbb{C}^3 \mid z_3 = 0\}$. Then $f|_{\mathbb{C}^3 \smallsetminus P}$ is a submersion, and hence open. Furthermore

$$\overline{R_{f|_{\mathbb{C}^3 \smallsetminus P}}} = R'.$$

Hence R' is the meromorphic equivalence relation associated to f, according to lemma 2.6.3. Furthermore, R_f is not a meromorphic equivalence relation.

2.6.5 Definition For a meromorphic equivalence relation R, the ***meromorphic quotient*** Φ_R of R is the set $\Phi_R := p_{1*}(Z(p_2)) \subset Z_*(X)$ with the induced topology. Elements of Φ_R are called ***meromorphic fibres***.

2.6.6 Remark By definition, $\Phi_R = p_{1*}\left(\overline{\mathrm{Im}(Z_{p_2,\mathrm{gen}})} \smallsetminus \{[\emptyset]\}\right)$. In [Sie92, Remark 8.5,1], Siebert shows that $\Phi_R = \overline{p_{1*}(\mathrm{Im}(Z_{p_2,\mathrm{gen}}))} \smallsetminus \{[\emptyset]\}$.

The following lemma describes the relation between $Z(p_2)$ and Φ_R:

2.6.7 Lemma *If R is a meromorphic equivalence relation, then the mapping*

$$(p_{1*}, \sigma) : Z(p_2) \longrightarrow \{(S,x) \in \Phi_R \times X \mid x \in |S|\},$$

where $\sigma : Z(p_2) \longrightarrow X$ is the base point mapping (see lemma 2.4.4), is a homeomorphism.

For the proof see [Sie92, Lemma 8.7].

We state now the theorem of Siebert-Grauert concerning meromorphic equivalence relations. This theorem is stated completely in [Sie92]. We present here only a part.

Let R be a meromorphic equivalence relation. We denote by $X' := (Z(p_2), {}_{Z(p_2)}\mathcal{O})$ the fibre-cycle space of $p_2 : R \longrightarrow X$. Consider the identification $X' = \{(S,x) \in \Phi_R \times X \mid x \in |S|\}$ of lemma 2.6.7.

2.6.8 Theorem of Grauert-Siebert *Let R be a meromorphic equivalence relation on a complex space X with polar set P and suppose that $X' := (Z(p_2), {}_{Z(p_2)}\mathcal{O})$ is a complex space. Then (see figure 2.6):*

(a) *The base point mapping $\sigma : X' \longrightarrow X$ is a proper modification*

(b) *The set*

$$R' := \{((S,x),(S',y)) \in X' \times X' \mid S = S'\}$$

is a geometrically flat equivalence relation on X'. The quotient space X'/R' is a maximal complex space, the canonical projection $q : X' \longrightarrow X'/R'$ is geometrically flat and

$$Z_q : X'/R' \longrightarrow Z(q)$$

is a homeomorphism.

(c) *There exists a homeomorphism from $Z(q)$ onto Φ_R. Thus there exists exactly one complex structure on Φ_R such that Φ_R is biholomorphic to X'/R'.*

For the proof see [Sie92, 8.8].

$$\begin{tikzcd}
R' \arrow[r] \arrow[d, "\tilde{p}_2"] & R \arrow[d, "p_2"] \\
X' \arrow[r, "\sigma"] \arrow[d, "q"] & X \\
X'/R' \arrow[r, "Z_q"] & Z(q) \cong \Phi_R
\end{tikzcd}$$

Figure 3: Mappings in the theorem of Grauert-Siebert

3 Singular holomorphic foliations

In this section, we present some elements in the theory of singular holomorphic foliations. We describe regular holomorphic foliations in a first subsection. The second subsection is dedicated to coherent holomorphic foliations and the third one to integrals.

3.1 Regular holomorphic foliations

In this subsection, we present the notion of regular holomorphic foliations. We fix some important notations for the following. At the end, we prove lemmas that will be useful.

Let X be a paracompact connected complex manifold of dimension n.

3.1.1 Definition A ***local regular holomorphic foliation*** of X is a surjective holomorphic submersion $f : U \longrightarrow V$ of an open subset U of X onto a manifold V.

We denote a local regular holomorphic foliation by (U, f, V).

3.1.2 Definition Two local regular holomorphic foliations (U_1, f_1, V_1) and (U_2, f_2, V_2) of X are called ***holomorphically compatible*** if for each $x \in U_1 \cap U_2$ there exist an open neighbourhood $W \Subset U_1 \cap U_2$ of x and a biholomorphic mapping $h : f_1(W) \longrightarrow f_2(W)$ such that $h \circ f_1 = f_2$.

3.1.3 Definition A ***regular holomorphic foliation*** $\mathbb{F}$ of dimension d of a connected paracompact complex manifold X is given by a maximal system $\mathcal{A}_{\mathbb{F}} := \{(U_j, f_j, V_j)\}$ of pairwise holomorphically compatible local regular holomorphic foliations on X such that $\bigcup U_j = X$ (maximal means that if (U, f, V) is a local regular holomorphic foliation that is holomorphically compatible with all the elements of $\mathcal{A}_{\mathbb{F}}$, then $(U, f, V) \in \mathcal{A}_{\mathbb{F}}$). The set $\mathcal{A}_{\mathbb{F}}$ is called the ***atlas*** of the foliation.
The ***codimension*** $\operatorname{codim} \mathbb{F}$ of $\mathbb{F}$ is the dimension of V where $(U, f, V) \in \mathcal{A}_{\mathbb{F}}$ and the ***dimension*** of $\mathbb{F}$ is given by

$$\dim \mathbb{F} = n - \operatorname{codim} \mathbb{F}.$$

A regular holomorphic foliation defines a topology on X called the ***leaf-topology***. A base of this topology is given by all the sets of the form $f^{-1}(f(x))$, where $(U, f, V) \in \mathcal{A}_{\mathbb{F}}$ and $x \in U$.

3.1.4 Definition A ***leaf*** of $\mathbb{F}$ is a connected component of X with respect to the leaf-topology.

For $x \in X$, L_x denotes the leaf through x. The foliation $\mathbb{F}$ defines an equivalence relation $R^{\mathbb{F}}$ given by $R^{\mathbb{F}}(x) = L_x$ for each $x \in X$. This is an open equivalence relation. We note $X/\mathbb{F} := X/R^{\mathbb{F}}$, and call it the ***leaf space*** of $\mathbb{F}$. The canonical projection $\pi : X \longrightarrow X/\mathbb{F}$ is an open mapping. If $A \subset X$ is $\mathbb{F}$-saturated, we note $A/\mathbb{F} := \pi(A)$.

3.1.5 Definition Let $\mathbb{F}$ be a regular foliation of dimension d. A couple (U, p), composed of a connected open subset U of X and a mapping $p : U \longrightarrow V$ where $V \Subset \mathbb{C}^{n-d}$, is called a ***local*** $\mathbb{F}$***-foliation*** if there exists a biholomorphic mapping $\alpha : U \longrightarrow V \times W$, where $W \Subset \mathbb{C}^d$ is connected, such that $(U, p, V) \in \mathcal{A}_{\mathbb{F}}$, the diagram

$$\begin{tikzcd} & U \arrow[dl, "p"'] \arrow[d, "\alpha"] \arrow[dr, "q"] & \\ V & V \times W \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & W \end{tikzcd}$$

commutes and there exists $(\widetilde{U}, \widetilde{p})$ with the same properties such that U is relatively compact in $\widetilde{U}$ and $p = \widetilde{p}|_U$.

If (U, p) is a local $\mathbb{F}$-foliation and $x \in U$, (U, p) is called a ***local*** $\mathbb{F}$***-foliation at*** $\boldsymbol{x}$.

3.1.6 Lemma *Let (U_1, p_1) and (U_2, p_2) be two local $\mathbb{F}$-foliations. If $x_1 \in U_1$ and $x_2 \in U_2$ belong to the same $\mathbb{F}$-leaf L, then there exist open neighbourhoods $\widetilde{V}_j \Subset V_j$ of $v_j := p_j(x_j)$ for $j = 1, 2$ and a biholomorphic mapping $h : \widetilde{V}_1 \longrightarrow \widetilde{V}_2$, such that the following holds:*

(a) $h(v_1) = v_2$

(b) *For each $v \in \widetilde{V}_1$, $p_2^{-1}(h(v))$ and $p_1^{-1}(v)$ belong to the same leaf.*

For the proof see [Hol78, Lemma 1.5].

3.1.7 Lemma *Each closed $\mathbb{F}$-leaf L is an analytic subset of X.*

For the proof see [Hol72, Theorem 3.1].

3.1.8 Lemma *The following conditions are equivalent:*

(a) *Each leaf of $\mathbb{F}$ is closed in X*

(b) *$X/\mathbb{F}$ is a T_1-space*

(c) *For each $L \in X/\mathbb{F}$, the set $\{L\}$ is closed in $X/\mathbb{F}$.*

Proof (b)⇒(c) is well-known. The implication (c)⇒(a) is a consequence of the continuity of π. It suffices to prove (a)⇒(b): let L_{x_1} and L_{x_2} be two different leaves of $\mathbb{F}$. Since X is paracompact, X is a regular topological space[5]. Thus, since L_{x_2} is closed in X, there exists an open neighbourhood $U \Subset X$ of x_1 such that $L_{x_2} \cap U = \emptyset$. Thus $L_{x_2} \cap R^{\mathbb{F}}(U) = \emptyset$, and hence $L_{x_2} \notin \pi(U)$. $\square$

3.1.9 Definition A regular holomorphic foliation $\mathbb{F}$ is called ***compact*** if each leaf of $\mathbb{F}$ is a compact subset of X.

By lemma 3.1.7, each leaf of a compact foliation is an analytic compact subset of X.

3.2 Singular holomorphic foliations

This subsection presents some results in the theory of singular holomorphic foliations. The principal references for this subsection are [BR85], [Rei97] and [HKR98]. In this subsection, X denotes a paracompact manifold of dimension n.

We recall the definition of singular holomorphic foliations given by Reiffen in [Rei97].

3.2.1 Definitions A ***representative of a singular holomorphic foliation*** $\mathbb{F}$ on X is a pair $(A, \mathbb{F}_A)$, where A is analytic of codimension at least 1 in X and $\mathbb{F}_A$ is a regular foliation on $X \setminus A$. Two representatives $(A, \mathbb{F}_A)$ and $(B, \mathbb{F}_B)$ are called equivalent if $\mathbb{F}_A|_{X \setminus (A\cup B)} = \mathbb{F}_B|_{X \setminus (A \cup B)}$. A ***singular holomorphic foliation*** of X is an equivalence class of representatives of holomorphic foliations. The ***singular locus*** $\operatorname{Sing}\mathbb{F}$ of $\mathbb{F}$ is given by

$$\operatorname{Sing}\mathbb{F} := \bigcap_{(A,\mathbb{F}_A)\in\mathbb{F}} A.$$

A singular holomorphic foliation $\mathbb{F}$ is called ***coherent*** if the equivalent conditions of proposition 3.2.2 are satisfied.

We adopt also the following notations

$$X^{\mathrm{reg}}(\mathbb{F}) := X \setminus \operatorname{Sing}\mathbb{F} \quad \text{and} \quad \mathbb{F}^{\mathrm{reg}} := \mathbb{F}|_{X^{\mathrm{reg}}(\mathbb{F})}.$$

We write X^{reg} if it is not necessary to precise the foliation.

We denote by ${}_X\Omega$ (or Ω) the sheaf of holomorphic Pfaffian forms on X and by ${}_X\Theta$ (or Θ) the sheaf of holomorphic vector fields on X. The sheaves Ω and Θ are free analytic[6] sheaves.

In [Rei97], Reiffen describes a method to associate to each singular holomorphic foliation $\mathbb{F}$ an analytic subsheaf $\Omega^{\mathbb{F}}$ of Ω and an analytic subsheaf $\Theta^{\mathbb{F}}$ of Θ (compare [Rei97, 3.11]).

[5] For topological definitions, one can refer to [Bou61] or [Eng89].

[6] A sheaf $\mathcal{F}$ on X is called **analytic** if it is an $\mathcal{O}$-module. A sheaf $\mathcal{F}$ is called free analytic if it is a free $\mathcal{O}$-module.

3.2.2 Proposition *For a singular holomorphic foliation* $\mathbb{F}$*, the following conditions are equivalent*

- $\operatorname{codim} \operatorname{Sing} \mathbb{F} \geq 2$
- $\Omega^{\mathbb{F}}$ *is coherent*
- $\Theta^{\mathbb{F}}$ *is coherent*

In his construction of $\Theta^{\mathbb{F}}$, resp. $\Omega^{\mathbb{F}}$, Reiffen proves that $\Theta^{\mathbb{F}}$, resp. $\Omega^{\mathbb{F}}$, is an involutive and complete subsheaf of Θ, resp. Ω. (See [Rei97, Definition 3.3] and [Rei97, Definition 1.1]). In [Rei97, Proposition 3.14], Reiffen proves that there exists a 1-1 correspondence between the set of coherent holomorphic foliations of dimension d on X and the set of involutive complete coherent analytic subsheaves of ${}_X\Omega$ (resp. ${}_X\Theta$) of rank d.

From now, we always suppose that all the singular holomorphic foliations in consideration are coherent.

As in the regular case, the notion of local leaf through x can be defined for certain points $x \in X$ (see [HKR99] for this construction or [Rei97] for a different but equivalent construction). In the singular case, there is not necessarily a local leaf through each x (see example 7.4.1). Define

$$X^\rho(\mathbb{F}) := \{x \in X \mid \text{there exists a local leaf through } x\} \quad \text{and} \quad \Sigma(\mathbb{F}) := X \smallsetminus X^\rho(\mathbb{F}).$$

We write X^ρ if it is not necessary to precise the foliation. In general, X^ρ is not open in X.

3.2.3 Definition The system of local leaves forms a base of the so-called ***leaf-topology*** on X^ρ, and a ***leaf*** is a connected component with respect to this topology. The leaves, with the leaf-topology, are complex spaces. We say that $\mathbb{F}$ ***has leaves everywhere*** if $\Sigma(\mathbb{F}) = \emptyset$.

It is possible to define an equivalence relation $R^{\mathbb{F}}$ on X^ρ as in the regular case: the equivalence classes of $R^{\mathbb{F}}$ are exactly the leaves of $\mathbb{F}$. We write $X^\rho/\mathbb{F} := X^\rho/R^{\mathbb{F}}$ and denote it the ***leaf space*** of $\mathbb{F}$. A subset A of X^ρ is called $\mathbb{F}$***-saturated*** if $R^{\mathbb{F}}(A) = A$. Contrary to the regular case, $R^{\mathbb{F}}$ is not open in general (see example 3.2.4).

3.2.4 Example of Rummler Let $\mathbb{F}$ be the foliation on $\mathbb{C}^4$ given by the mapping $f : \mathbb{C}^4 \longrightarrow \mathbb{C}^3$ defined by

$$\begin{aligned} f(z) &:= \left(z_1, z_2, \det \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) \\ &= (z_1, z_2, z_1 z_4 - z_3 z_2). \end{aligned}$$

One sees that

$$\operatorname{Sg} f = \operatorname{Sing} f = \{z \in \mathbb{C}^4 \mid \operatorname{rank} d_z f < 3\} = \{(0,0)\} \times \mathbb{C}^2.$$

Since the fibers of f are affine subspaces of $\mathbb{C}^4$, f is simple. Reiffen in [Rei97, Example 7.6] proves that f is an integral of $\mathbb{F}$ (see definition 3.3.1), $\mathbb{F}$ has leaves everywhere and the leaves of $\mathbb{F}$ are exactly the fibers of f. Hence $R^{\mathbb{F}}$ is the equivalence relation on $\mathbb{C}^4$ given by f. One sees that

$$\operatorname{Im} f = (\mathbb{C}^3 \setminus (\{(0,0)\} \times \mathbb{C})) \cup \{(0,0,0)\}.$$

Since $f^{-1}(0,0,0) = \{(0,0)\} \times \mathbb{C}^2$ is of dimension 2 and for each $(c_1, c_2) \neq (0,0)$ and $d \in \mathbb{C}$, $f^{-1}(c_1, c_2, d)$ has dimension 1, f is not open. Hence $R^{\mathbb{F}}$ is not open.

3.2.5 Definition A coherent holomorphic foliation $\mathbb{F}$ is called ***open*** if the following conditions are satisfied:

(a) $R^{\mathbb{F}}$ is open

(b) for each open subset $U \Subset X$ such that $U \subset X^{\rho}$, the saturation $R^{\mathbb{F}}(U)$ of U is open in X.

3.2.6 Remark If X^{ρ} is open in X, then in the previous definition (a)$\Longleftrightarrow$(b).

An $\mathbb{F}$-leaf L is called ***proper*** if it is an analytic subset of X. The foliation is called ***proper*** if it has leaves everywhere and if each leaf is proper.

3.2.7 Remark Each regular foliation with all leaves closed is proper (by lemma 3.1.7).

3.2.8 Definition A singular holomorphic foliation $\mathbb{F}$ is called ***compact*** if it is proper and if each leaf of $\mathbb{F}$ is a compact subset of X.

3.2.9 Definition The ***singular hull*** $\operatorname{Sh}\mathbb{F}$ of $\mathbb{F}$ is given by

$$\operatorname{Sh}\mathbb{F} := \operatorname{Sing}\mathbb{F} \cup R^{\mathbb{F}}(X^{\rho} \cap \operatorname{Sing}\mathbb{F}),$$

The foliation $\mathbb{F}$ is called ***generically regular*** if $\operatorname{Sh}\mathbb{F}$ is analytically thin in X.

3.2.10 Lemma *The set $X \setminus \operatorname{Sh}\mathbb{F}$ is included in X^{ρ} and $\mathbb{F}$-saturated. Furthermore, if $\mathbb{F}$ is open, then the set $X \setminus \overline{\operatorname{Sh}\mathbb{F}}$ is $\mathbb{F}$-saturated.*

Proof The inclusion holds by definition. If $x \in X \setminus \operatorname{Sh}\mathbb{F}$, then $L_x \cap \operatorname{Sing}\mathbb{F} = \emptyset$. Hence, $L_x \subset X \setminus \operatorname{Sh}\mathbb{F}$.
We prove that $U := X \setminus \overline{\operatorname{Sh}\mathbb{F}}$ is $\mathbb{F}$-saturated. Since $X \setminus \operatorname{Sh}\mathbb{F}$ is $\mathbb{F}$-saturated,

$$U \subset R^{\mathbb{F}}(U) \subset X \setminus \operatorname{Sh}\mathbb{F}.$$

Furthermore, since $\mathbb{F}$ is open and U is open in X, $R^{\mathbb{F}}(U)$ is open in X. Since U is the biggest open subset of X such that $U \subset X \setminus \operatorname{Sh}\mathbb{F}$, $U = R^{\mathbb{F}}(U)$, i.e. U is $\mathbb{F}$-saturated. $\square$

3.2.11 Remark If $\mathbb{F}$ has leaves everywhere, then the definition of $\mathrm{Sh}\mathbb{F}$ given in [HKR98] coincides with definition 3.2.9.

Define

$$\begin{array}{rcl} X^{\mathrm{ns}}(\mathbb{F}) & := & X \setminus \overline{\mathrm{Sh}\mathbb{F}} \\ \mathbb{F}^{\mathrm{ns}} & := & \mathbb{F}|_{X^{\mathrm{ns}}}. \end{array}$$

We write X^{ns} if it is not necessary to precise the foliation. The foliation $\mathbb{F}^{\mathrm{ns}}$ is regular and the leaves of $\mathbb{F}^{\mathrm{ns}}$ are leaves of F.

3.2.12 Remark The set X^{ns} is open in X and included in X^{reg}. Furthermore if $\mathbb{F}$ is generically regular, then X^{ns} is dense in X, because $\overline{\mathrm{Sh}\mathbb{F}}$ is nowhere dense in X. If $\mathbb{F}$ is open, then the set X^{ns} is a union of $\mathbb{F}$-leaves (see lemma 3.2.10).

3.3 Integrals

This subsection presents the notion of integrals. We present some interesting results that connect fibres of global integrals and leaves of foliations. The principal reference for this subsection is [Rei97]. In this subsection, X denotes a paracompact connected complex manifold of dimension n.

3.3.1 Definition If $\mathbb{F}$ is a singular holomorphic foliation on X, then a generically open holomorphic mapping $f\colon U \longrightarrow Y$ from an open subset U of X to an irreducible complex space Y is called an ***integral*** of $\mathbb{F}$ if

$$\Omega^{\mathbb{F}}|_U = \widetilde{f^*{}_Y\Omega},$$

where $f^*{}_Y\Omega$ denotes the analytic subsheaf of $\Omega = {}_X\Omega$ generated by the Pfaffian forms $f^*\omega$, $\omega \in {}_Y\Omega$. For an analytic sheaf $\mathcal{F}$, $\widetilde{\mathcal{F}}$ denotes the completion of the sheaf $\mathcal{F}$ (see [Rei97, Definition 1.1]).

We say that an integral $f\colon U \longrightarrow Y$ is ***global*** if $U = X$.

3.3.2 Lemma *If $f\colon X \longrightarrow Y$ is a global integral of $\mathbb{F}$, then*

$$f|_{X\setminus \mathrm{Sg}\, f}\colon X \setminus \mathrm{Sg} \longrightarrow f(X \setminus \mathrm{Sg}\, f)$$

is a submersion and $(X \setminus \mathrm{Sg}\, f, f|_{X\setminus \mathrm{Sg}\, f}, f(X \setminus \mathrm{Sg}\, f))$ is a local regular foliation of $\mathbb{F}$. (compare definition 3.1.3).

Proof We know that $f|_{X\setminus \mathrm{Sg}\, f}$ is a submersion. Furthermore

$$\Omega^{\mathbb{F}}|_{X\setminus \mathrm{Sg}\, f} = (\widetilde{f^*{}_Y\Omega})|_{X\setminus \mathrm{Sg}\, f} = (f^*{}_Y\Omega)|_{X\setminus \mathrm{Sg}\, f}$$

is a regular subsheaf of ${}_X\Omega|_{X\setminus \mathrm{Sg}\, f}$ (see [Rei97, Definition 1.5]). Thus $f|_{X\setminus \mathrm{Sg}\, f}$ is a local regular foliation of $\mathbb{F}^{\mathrm{reg}}$. $\square$

3.3.3 Lemma *If $f : X \longrightarrow Y$ is a global integral of $\mathbb{F}$, then $L_x \subset f^{-1}(f(x))$ for each $x \in X^\rho$.*

Proof Let $x \in X^\rho$. We know that $f^*{}_Y\Omega$ is a subsheaf of $\Omega^{\mathbb{F}} = \widetilde{f^*{}_Y\Omega}$. Thus, by [Rei97, Definition 3.30, Proposition 6.9 and Remark 6.5], $f \circ i$ is constant, where $i : L_x \longrightarrow X$ is the canonical inclusion. It follows that $f(y) = f(x)$ for each $y \in L_x$. $\square$

3.3.4 Lemma *If $\mathbb{F}$ is a regular foliation and if $f : X \longrightarrow Y$ is a global integral of $\mathbb{F}$ such that Y is a manifold, then* $\operatorname{Sing} f$ *is $\mathbb{F}$-saturated.*

Proof Let $x \in \operatorname{Sing} f$.
We will first show that there exists a local leaf through x that is included in $\operatorname{Sing} f$. Let (U, p) be a local $\mathbb{F}$-foliation at x. We use notations defined in definition 3.1.5. We can suppose that $U = \alpha(U)$, $x = 0$, $p(z) = (z_{d+1}, \ldots, z_n)$ and $q(z) = (z_1, \ldots, z_d)$. The fact that $0 \in \operatorname{Sing} f$ is equivalent to the fact that $d_0 f$ does not have maximal rank. By lemma 3.3.3, $L_z \subset f^{-1}(f(z))$ for each $z \in U$. Thus $f(z) = f(0, \ldots, 0, z_{d+1}, \ldots, z_n)$ for each $z \in U$. Hence,

$$\begin{array}{rcll} \dfrac{\partial f}{\partial z_k}(z) & = & 0 & \forall\ 1 \leq k \leq d \\ \dfrac{\partial f}{\partial z_k}(z) & = & \dfrac{\partial f}{\partial z_k}(0, \ldots, 0, z_{d+1}, \ldots, z_n) & \forall\ d+1 \leq k \leq n. \end{array}$$

Thus $d_{(z_1, \ldots, z_d, 0, \ldots, 0)} f = d_0 f$ and it has not maximal rank, i.e. $p^{-1}(0) \subset \operatorname{Sing} f$.
Now let $y \in L_x$. We take a path γ included in L_x from x to y and cover it with a finite number of local $\mathbb{F}$-foliation. By the above argumentation, it follows that $y \in \operatorname{Sing} f$. $\square$

Stability, multiplicities and cycles

4 Some general theories

4.1 The sets $\mathcal{H}_1(T)$ and $\mathcal{H}_2(T)$

We present in this subsection a local definition of Hausdorff spaces.

In this subsection, T denotes a topological space.

4.1.1 Definition For a topological space T we define

$$\mathcal{H}_1(T) := \{x \in T \mid \text{there exists an open neighbourhood } U \Subset T \text{ of } x \text{ such that } \overline{U} \text{ is Hausdorff}\}$$

and

$$\mathcal{H}_2(T) := \{x \in T \mid \text{for each } y \in T \smallsetminus \{x\}, \text{ there exist open neighbourhoods } U_x \text{ and } U_y \text{ of } x, \text{ resp. } y, \text{ in } X \text{ such that } U_x \cap U_y = \emptyset\}.$$

4.1.2 Proposition *$\mathcal{H}_1(T)$ is open in T and $\mathcal{H}_1(T) \subset \mathcal{H}_2(T)$.*

Proof $\mathcal{H}_1(T)$ is open by definition. Let $x \in \mathcal{H}_1(T)$ and let U be an open neighbourhood of x such that $\overline{U}$ is Hausdorff. Let $y \in X$ different from x. If $y \in \overline{U}$, then there exist an open neighbourhood U' of x in X and an open neighbourhood V of y in X such that $(U' \cap \overline{U}) \cap (V \cap \overline{U}) = \emptyset$. Hence $(U' \cap U) \cap V = \emptyset$. If $y \notin \overline{U}$, then $U \cap (X \smallsetminus \overline{U}) = \emptyset$. Hence $x \in \mathcal{H}_2(T)$. □

4.1.3 Remarks The inclusion $\mathcal{H}_2(T) \subset \mathcal{H}_1(T)$ is false in general, as it is shown by example 4.1.4. This example shows also that in general $\mathcal{H}_2(T)$ is not open in T.

4.1.4 Example Let $T' := \mathbb{R}$, $T'' := \mathbb{R}^*$ and $T := T' \amalg T''$. Let $\varphi : T'' \longrightarrow T'$ be the canonical injection of $\mathbb{R}^*$ in $\mathbb{R}$. For $x \in T$, we define

$$\mathcal{B}(x) := \begin{cases} \{U \Subset T' \mid x \in U\} & \text{if } x \in T' \\ \Big\{(U \smallsetminus \{\varphi(x)\}) \amalg \{x\} \;\Big|\; \varphi(x) \in U \Subset T'\Big\} & \text{if } x \in T''. \end{cases}$$

This is a base of neighbourhoods of x. These bases define a T_1-topology on T. If $x \in T''$, the points x and $\varphi(x)$ could not be separated by open sets. Hence T is not Hausdorff. This shows also that $\mathcal{H}_2(T) = \{0\} \subset T'$ and for all $V \Subset T$, the closure $\overline{V}$ is not Hausdorff, i.e. $\mathcal{H}_1(T) = \emptyset$.

4.1.5 Proposition *The subsets $\mathcal{H}_1(T)$ and $\mathcal{H}_2(T)$ of T are Hausdorff spaces (with the induced topology)*

Proof *$\mathcal{H}_1(T)$ is Hausdorff:* By proposition 4.1.2, $\mathcal{H}_1(T) \subset \mathcal{H}_2(T)$. Hence if x, $y \in \mathcal{H}_1(T)$ and $x \neq y$, then there exists an open neighbourhood $U_x \Subset T$ of x and an open neighbourhood $U_y \Subset T$ of y such that $U_x \cap U_y = \emptyset$. Hence

$$(\mathcal{H}_1(T) \cap U_x) \cap (\mathcal{H}_1(T) \cap U_y) = \emptyset.$$

$\mathcal{H}_2(T)$ is Hausdorff: Let $x_1 \neq x_2 \in \mathcal{H}_2(T)$. By definition of $\mathcal{H}_2(T)$, there exists an open neighbourhood $U_j \Subset T$ of x_j, $j = 1, 2$, such that $U_1 \cap U_2 = \emptyset$. Hence $(U_1 \cap \mathcal{H}_2(T)) \cap (U_2 \cap \mathcal{H}_2(T)) = \emptyset$, which completes the proof. □

4.1.6 Theorem *For a topological space T, the following conditions are equivalent:*

(a) *T is Hausdorff*

(b) $\mathcal{H}_1(T) = T$

(c) $\mathcal{H}_2(T) = T$

Proof For (a)⇒(b) we choose $U = T$ in the definition of $\mathcal{H}_1(T)$. (b)⇒(c) follows from proposition 4.1.2. (c)⇒(a) follows from the definition of $\mathcal{H}_2(T)$. □

For a better comprehension of these two subsets, it would be interesting to speak about maximal Hausdorff open subspaces.

4.1.7 Definition An open subset $U \Subset T$ is called a ***maximal Hausdorff open subspace*** of T if the following conditions are satisfied:

i) U is Hausdorff (with the induced topology)

ii) if $V \Subset T$ is an open Hausdorff subspace with $U \subset V$ then $U = V$.

4.1.8 Lemma *If T is a topological space, then there exists a maximal Hausdorff open subspace of T.*

Proof Let $\mathfrak{U} := \{U \Subset T \mid U \text{ Hausdorff}\}$. $\mathfrak{U}$ is partially ordered by $\subset$. Let $\mathcal{L} \subset \mathfrak{U}$ be a totally ordered subset of $\mathfrak{U}$, and $S_{\mathcal{L}} := \bigcup_{U \in \mathcal{L}} U$. This set is a bound of $\mathcal{L}$.
We prove that $S_{\mathcal{L}}$ is Hausdorff. Let $x_1 \neq x_2 \in S_{\mathcal{L}}$. By definition of $S_{\mathcal{L}}$, there exist $U_j \in \mathcal{L}$ such that U_j is a neighbourhood of x_j, $j = 1, 2$. Since $\mathcal{L}$ is totally ordered, we assume that $U_1 \subset U_2$. Hence x_1 and x_2 can be separated by open subsets of U_2. Thus $S_{\mathcal{L}}$ is Hausdorff. Thus, by Zorn's Lemma, $\mathfrak{U}$ has a maximal element. □

4.1.9 Proposition *If U is a maximal Hausdorff open subspace of T, then $\mathcal{H}_1(T) \subset U$. In addition, if $\mathcal{H}_2(T)$ is open in T, then $\mathcal{H}_2(T) \subset U$.*

Proof Let $V := \mathcal{H}_1(T) \cup U$.
We will prove that V is Hausdorff. Let $x \neq y \in V$. If $x, y \in \mathcal{H}_1(T)$ or if $x, y \in U$, they can be separated by open subsets of T. If $x \in \mathcal{H}_1(T)$ and $y \in U$, then there exist an open neighbourhood $V_x \Subset T$ of x and an open neighbourhood $V_y \Subset T$ of y such that $V_x \cap V_y = \emptyset$ ($x \in \mathcal{H}_2(T)$ by proposition 4.1.2). Hence

$$(V \cap V_x) \cap (V \cap V_y) = \emptyset.$$

This proves that V is Hausdorff. Since U is a maximal open subspace of T and $U \subset V$, then $V = U$ and finally $\mathcal{H}_1(T) \subset U$.
The proof of $\mathcal{H}_2(T) \subset U$ is similar: it suffices to prove that $V := \mathcal{H}_2(T) \cup U$ is Hausdorff. Let $x \neq y \in V$. If $x, y \in \mathcal{H}_2(T)$ or $x, y \in U$, they can be separated by open subsets of T, since $\mathcal{H}_2(T)$ and U are Hausdorff. If $x \in \mathcal{H}_2(T)$ and $y \in U$, they also can be separated by open subsets of T, by the definition of $\mathcal{H}_2(T)$. □

In general, there exist several maximal Hausdorff open spaces. Furthermore, $\mathcal{H}_1(T)$ is in general not a maximal open Hausdorff space.

4.1.10 Example Let $T := (\mathbb{C} \times \{0\}) \cup (\mathbb{C} \times \{1\}) \subset \mathbb{C}^2$ and R be the equivalence relation on T given by

$$R(x, j) := \begin{cases} \{(x,0),(x,1)\} & \text{if } x \neq 0 \\ \{(x,j)\} & \text{if } x = 0, \end{cases}$$

where $j = 0, 1$. We denote $T' := T \setminus \{(0,0),(0,1)\}$. Then the maximal Hausdorff open spaces of T/R are

$$(T' \cup \{(0,0)\})/R \quad \text{and} \quad (T' \cup \{(0,1)\})/R.$$

Furthermore, $\mathcal{H}_1(T/R) = T'/R$, which is not a maximal Hausdorff open space.

If T is not Hausdorff, the quasi-compactness[7] of a subset $K \subset T$ implies generally not that K is closed in T. But, for a point in $\mathcal{H}_2(T)$ we have the following

4.1.11 Lemma *Let $x \in \mathcal{H}_2(T)$ and $K \subset T$ be quasi-compact such that $x \notin K$. Then there exists an open neighbourhood $W \Subset T$ of x such that $W \cap K = \emptyset$.*

Proof For each $y \in K$, there exist an open neighbourhood $U_y \Subset T$ of y and an open neighbourhood $V_y \Subset T$ of x such that $U_y \cap V_y = \emptyset$. By construction $K \subset \bigcup_{y \in K} U_y$. Thus there exist $y_1, \ldots, y_N \in K$ such that $K \subset \bigcup_{j=1}^N U_{y_j}$. Let $W := \bigcap_{j=1}^N V_{y_j}$. This is an open neighbourhood of x. Furthermore $W \cap \left(\bigcup_{j=1}^N U_{y_j}\right) = \emptyset$. Hence $W \cap K = \emptyset$. □

[7] We recall that a topological space K is called **quasi-compact** if each open cover of K has a finite subcover. With that terminology, K is compact if it is quasi-compact and Hausdorff

4.2 Quasi-finite equivalence relations and R-multiplicity

In this section, T denotes a Hausdorff topological space.

Let R be a quasi-finite equivalence relation on T. The function $\nu_R : T \longrightarrow \mathbb{N}_{>0}$ is defined by $\nu_R(x) := \operatorname{Card} R(x)$.

4.2.1 Definitions A quasi-finite equivalence relation R on T is called ***bounded*** if ν_R is bounded, i.e. if there exists $M \in \mathbb{N}_{>0}$ such that $\nu_R(x) \le M$ for all $x \in T$.
A quasi-finite equivalence relation R on T is called ***bounded in*** $\boldsymbol{x_0 \in T}$ if there exists an open neighbourhood $U \Subset T$ of x_0 such that the equivalence relation $R|_U$ on U is bounded.

4.2.2 Example Let $T := \{x \in \mathbb{C}^2 \mid |x_1| < 1,\ |x_2| < 1\}$. We define the equivalence relation R on T by

$$R(x) := \{y \in T \mid y_2 = x_2,\ x_1 - y_1 \in \mathbb{Z}x_2\}.$$

This equivalence relation is quasi-finite, but not bounded in points $(z, 0)$.

If R is bounded in $x \in T$ and if U is an open neighbourhood of x such that $R|_U$ is bounded, then the maximum

$$M_R(U) := \max_{v \in U} \left(\nu_{R|_U}(v)\right)$$

is a well-defined natural number. In addition, if $U' \Subset U$ is a smaller open neighbourhood of x, then $M_R(U') \le M_R(U)$.

4.2.3 Definition If R is bounded in $x \in T$, the ***R-multiplicity*** $\mu_R(x)$ of x is defined by

$$\mu_R(x) := \min_{x \in U \Subset T} \left(M_R(U)\right) = \min_{x \in U \Subset T} \Big(\max_{v \in U} \left(\nu_{R|_U}(v)\right) \Big).$$

4.2.4 Proposition *The number $\mu_R(x)$ has the following properties:*

(a) *There exists an open neighbourhood $U \Subset T$ of x such that $\mu_R(x) = M_R(U)$.*

(b) *If U is an open neighbourhood of x, then $\mu_{R|_U}(x) = \mu_R(x)$.*

(c) *If there exists an open neighbourhood $U \Subset T$ of x such that $\nu_R(u) = 1$ for each $u \in U$, then $\mu_R(x) = 1$.*

Proof (a) is clear. We prove (b). First, for each open neighbourhood $V \Subset T$ of x,

$$M_{R|_U}(V \cap U) = \max_{v \in V \cap U} \big(\underbrace{\nu_{R|_{U \cap V}}(v)}_{\le \nu_{R|_V}(v)} \big) \le M_R(V).$$

Hence

$$\mu_{R|_U}(x) = \min_{x\in V\Subset U}(M_{R|_U}(V)) = \min_{x\in V\Subset T}(M_{R|_U}(V\cap U)) \leq \min_{x\in V\Subset T}(M_R(V)) = \mu_R(x).$$

Moreover,

$$\mu_R(x) = \min_{x\in V\Subset T}(M_R(V)) \leq M_R(U') = M_{R_U}(U') \quad \text{for each } U'\Subset U \text{ with } x\in U'.$$

The second inequality proves that $\mu_R(x) \leq \mu_{R|_U}(x)$. For (c),

$$\mu_R(x) = \min_{x\in V\Subset U}(M_R(V)) \leq M_R(U) \leq \max_{u\in U}\nu_R(u) = 1.$$

□

To the end of this section X denotes a paracompact complex space. We present here some results of [Hol78] and some new results in connection with the R-multiplicity.

4.2.5 Definition An equivalence relation R on X is called ***weakly-analytic*** if through each $(x, y) \in R$ passes a set $A \subset R$ which is local analytic in $X \times X$, and is mapped biholomorphically onto an open neighbourhood of x in X by the projection on the first factor and onto an open neighbourhood of y in X by the projection on the second factor.

The following proposition is due to Holmann (compare [Hol78, Lemma 2.1])

4.2.6 Proposition *If R is an open, quasi-finite and weakly-analytic equivalence relation on X, then*

(a) *ν_R is semi-continuous from below (i.e. for each $x \in X$ there exists an open neighbourhood $U \Subset X$ of x such that $\nu_R(x) \leq \nu_R(y)$ for all $y \in U$);*

(b) *$C_R := \{x \in X \mid \nu_R$ is continuous in $x\}$ is open and dense in X.*

(c) *$C_R \subset \{x \in X \mid R$ is bounded in $x\}$.*

Proof Ad (a). Since R is weakly-analytic, there exists for each $x \in X$ an open neighbourhood $U \Subset X$ of x such that $\nu_R(y) - \nu_R(x) \geq 0$ for each $y \in U$.
Ad (b). If ν_R is continuous in $x \in V$, then there exists an open neighbourhood $U \Subset X$ of x such that $\nu_R(x) = \nu_R(y)$ for each $y \in U$, i.e. ν_R is continuous on a neighbourhood of x.
To prove the density of C_R we suppose that this is not the case. Since C_R is open, there exists an open subset $U \Subset X$ such that ν_R is discontinuous in each point of U. Since ν_R is semi-continuous from below, the set

$$A_n := \{u \in U \mid \nu_R(u) \leq n\}$$

is closed in U for each n. Since R is quasi-finite, $\bigcup_{n\in\mathbb{N}} A_n = U$. Let n_0 be the minimum of $m \in \mathbb{N}$ such that $\mathring{A}_m \neq \emptyset$. This minimum exists by Baire's theorem.

Let $U' \Subset U$ such that $U' \subset A_{n_0}$. We define $U'' := U' \setminus (A_{n_0-1} \cap U')$. The set U'' is open in U and non-void. By construction, $\nu_R(u) = n_0$ for each $u \in U''$, i.e. ν_R is continuous on U'' which is a contradiction to the assumption.
Ad (c). If $x \in C_R$, then there exists an open neighbourhood $U \Subset X$ of x such that $\nu_R(x) = \nu_R(y)$ for each $y \in U$. Thus for each $y \in U$,

$$\nu_{R|_U}(y) \leq \nu_R(y) = \nu_R(x).$$

Hence $R|_U$ is bounded. □

4.2.7 Theorem *Let R be an open, quasi-finite and weakly-analytic equivalence relation on X. Then the set*

$$X_{\mathrm{tr}}^R := \{x \in X \mid \mu_R(x) = 1\}$$

is open and dense in X.

For the proof of this theorem, we need

4.2.8 Lemma *If R is an open, quasi-finite and weakly-analytic equivalence relation on X, then there exists $x \in X$ such that $\mu_R(x) = 1$.*

Proof By proposition 4.2.6 the set C_R is open and dense in X.
If there exists $x \in C_R$ such that $\nu_R(x) = 1$, then there exists an open neighbourhood $U \Subset X$ of x such that $\nu_R(u) = 1$ for each $u \in U$, and thus $\mu_R(x) = 1$ by proposition 4.2.4(c). This completes the proof.
Hence we suppose that $\nu_R(x) \neq 1$ for all $x \in C_R$. Let $x_1 \in C_R$ and let $U_1 \Subset C_R$ be an open, connected and relatively compact neighbourhood of x_1 such that $R(x_1) \cap U_1 = \{x_1\}$. Since U_1 is connected and ν_R is continuous on U_1,

$$\nu_R(x) = \nu_R(x_1) =: M_1 \quad \text{for all } x \in U_1.$$

Let $x' \in R(x_1)$ be different from x_1. Then there exists an open neighbourhood $V_1 \Subset R(U_1)$ of x'. Thus, for each $y \in V_1$, $\nu_R(y) = M_1$, i.e. ν_R is constant on V_1. Thus $V_1 \subset C_R$. We restrict U_1 and V_1 such that $R(x') \cap V_1 = \{x'\}$, $U_1 \cap V_1 = \emptyset$ and $U_1 \Subset R(V_1)$ (with these restrictions, $V_1 \not\subset R(U_1)$ in general). We define

$$R_1 := R|_{U_1} = R \cap (U_1 \times U_1).$$

This equivalence relation is weakly-analytic, open and quasi-finite. By proposition 4.2.6 the set C_{R_1} is open and dense in U_1. Let $x_2 \in C_{R_1}$ and $U_2 \Subset C_{R_1}$ be an open, connected and relatively compact neighbourhood of x_2 such that $R_1(x_2) \cap U_2 = \{x_2\}$. Hence

$$\nu_{R_1}(y) = \nu_{R_1}(x_2) =: M_2 \quad \text{for all } y \in U_2.$$

Thus

$$M_2 = \nu_{R_1}(x_2) = \operatorname{Card} R_1(x_2) = \operatorname{Card}(R(x_2) \cap U_1) \underset{\substack{\uparrow \\ \text{by the choice of } U_1}}{<} \operatorname{Card} R(x_2) = \nu_R(x_2) = M_1.$$

If $M_2 = 1$, then $\mu_{R_1}(x_2) = 1$ by proposition 4.2.4(c) and $\mu_R(x_2) = 1$ by proposition 4.2.4(b). Hence the proof is completes.
If $M_2 \neq 1$, we repeat the construction. Thus we obtain a sequence

$$M_1 > M_2 > \cdots > M_N = 1.$$

By the same argumentation, we calculate

$$\mu_R(x_N) = \mu_{R_{N-1}}(x_N) = 1,$$

which concludes the proof. $\square$

Proof of theorem 4.2.7 Let $x \in X^R_{\mathrm{tr}}$. By proposition 4.2.4(a) there exists an open neighbourhood $U \Subset X$ of x such that

$$\mu_R(x) = M_R(U) = \max_{u \in U} (\nu_{R|_U}(u)),$$

i.e. $\nu_{R|_U}(u) = 1$ for each $u \in U$. Hence for each $y \in U$

$$\mu_R(y) = \min_{y \in U' \Subset X} (M_R(U')) \leq M_R(U) = 1,$$

and thus $\mu_R(y) = 1$, which proves that X^R_{tr} is open in X.
For the proof that X^R_{tr} is dense in X , let $U \Subset X$. Since $R|_U$ is open, quasi-finite and weakly-analytic, there exists $x \in U$ such that $\mu_{R|_U}(x) = 1$ by lemma 4.2.8. By proposition 4.2.4(b), $\mu_R(x) = 1$, i.e. $x \in X^R_{\mathrm{tr}}$. $\square$

4.2.9 Lemma *Let R be an open, quasi-finite and weakly-analytic equivalence relation on X. Suppose that R is bounded in $x \in X$. By proposition 4.2.4(a) we can find an open neighbourhood U of x such that $\mu_R(x) = M_R(U)$. Then the set*

$$\{u \in U \mid \nu_R(u) = \mu_R(x)\}$$

is open in X and included in X^R_{tr}.

Proof Define $E := \{u \in U \mid \nu_R(u) = \mu_R(x)\}$. Because of the semi-continuity of ν_R, the set $\{u \in U \mid \nu_R(u) \leq \mu_R(x) - 1\} = U \setminus E$ is closed in U, i.e. E is open in U.
We prove that $E \subset X^R_{\mathrm{tr}}$. Let $u_1 \in E$. Since E is open in U, there exists an open neighbourhood $\widetilde{U} \Subset U$ of u_1 such that $\nu_R(u) = \mu_R(x)$ for each $u \in \widetilde{U}$. We denote

$$R(u_1) =: \{u_1, \ldots, u_{\mu_R(x)}\}.$$

There exist $\mu_R(x)$ pairwise disjointed open subsets $U_j \Subset \widetilde{U}$ such that, for each $j = 1, \ldots, \mu_R(x)$, U_j is a neighbourhood of u_j and $\nu_R(u) = \mu_R(x)$ for each $u \in U_j$. We restrict U_1 such that $U_1 \subset R(U_j)$ for each $j = 2, \ldots, \mu_R(x)$. If $\widetilde{u} \in U_1$, then, for each $j = 2, \ldots, \mu_R(x)$, $\widetilde{u} \in R(U_j)$, i.e. there exists $\widetilde{u}_j \in U_j \cap R(\widetilde{u})$. Thus $R(\widetilde{u}) \supset \{\widetilde{u}, \widetilde{u}_2, \ldots, \widetilde{u}_{\mu_R(x)}\}$. Since $\nu_R(\widetilde{u}) = \mu_R(x)$, $R(\widetilde{u}) = \{\widetilde{u}, \widetilde{u}_2, \ldots, \widetilde{u}_{\mu_R(x)}\}$, and thus $R(\widetilde{u}) \cap U_1 = \{\widetilde{u}\}$. Thus $\nu_{R|_{U_1}}(\widetilde{u}) = 1$ for each $\widetilde{u} \in U_1$. i.e. $M_R(U_1) = 1$. Hence $\mu_R(u_1) = 1$, i.e. $u_1 \in X^R_{\mathrm{tr}}$. $\square$

The set $\{u \in U \mid \nu_R(u) = \mu_R(x)\}$ is in general not dense in U, as it is shown by example 4.2.10.

4.2.10 Example Let $X := \mathbb{C}$ and let R be the equivalence relation on X given by $R(z) := \{z, -z\}$. It is a finite, open and analytic equivalence relation. But $\mu_R(z) = 1$ for each $z \in X \setminus \{0\}$. For $z \in X \setminus \{0\}$ fixed, such that $\operatorname{Re} z > 0$, the open subset $U := \{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$ of $\mathbb{C}$ is a neighbourhood of z. Furthermore $\nu_R(z) = M_R(U) = 1$ and

$$\{x \in U \mid \nu_R(x) = \mu_R(z)\} = \emptyset,$$

since $\nu_R(x) = 2$ for each $x \in U$.

4.3 The Hausdorff metric and proper equivalence relations

In this subsection, we recall the notion of the Hausdorff metric on the set $\mathcal{K}(X)$ of compact subsets of a locally compact metric space X. We prove that the topology on $\mathcal{K}(X)$ associated to the Hausdorff metric is independent of the choice of the metric on X (compare theorem 4.3.3). At the end, we prove the relation between the fact that an equivalence relation is proper and the fact that the canonical mapping from X to the set of compact subsets of X is continuous (compare theorem 4.3.8).

For a locally compact metric space (X, d), denote

$$\mathcal{K}(X) := \{A \subset X \mid A \text{ is compact and } A \neq \emptyset\}.$$

If (X, d) is a locally compact metric space, then, for each $K, L \in \mathcal{K}(X)$, we use the following notations

$$\begin{aligned} d(x, L) &:= \min\{d(x, y) \mid y \in L\} \\ d(K, L) &:= \min\{d(x, L) \mid x \in K\} \\ d_K(L) &:= \max\{d(x, L) \mid x \in K\} \end{aligned}$$

4.3.1 Definition If (X, d) is a locally compact metric space, then the ***Hausdorff metric*** $d_{\mathcal{H}}$ on $\mathcal{K}(X)$ is defined by

$$d_{\mathcal{H}}(K, L) := \max\{d_K(L), d_L(K)\}.$$

By [Bar88, §6.2], $d_{\mathcal{H}}$ is a metric on $\mathcal{K}(X)$.

For $K \in \mathcal{K}(X)$ and $\varepsilon > 0$, denote

$$\begin{aligned} U_\varepsilon(K) &:= \{x \in X \mid d(x, K) < \varepsilon\} \Subset X \\ B_\varepsilon^{\mathcal{H}}(K) &:= \{L \in \mathcal{K}(X) \mid d_{\mathcal{H}}(K, L) < \varepsilon\} \Subset \mathcal{K}(X). \end{aligned}$$

4.3.2 Lemma *For a locally compact metric space* (X, d),

$$d_{\mathcal{H}}(K, L) < \varepsilon \iff K \subset U_\varepsilon(L) \text{ and } L \subset U_\varepsilon(K).$$

For the proof, see [Bar88, Lemma 1, §2.7]

4.3.3 Theorem *Let X be a locally compact metrizable topological space. If d and $\widetilde{d}$ are two metrics that generate the topology of X, then, denoting $d_{\mathcal{H}}$ and $\widetilde{d}_{\mathcal{H}}$ the associated Hausdorff metrics, the mapping $(\mathcal{K}(X), d_{\mathcal{H}}) \xrightarrow{\mathrm{Id}} (\mathcal{K}(X), \widetilde{d}_{\mathcal{H}})$ is a homeomorphism.*

Proof We have to prove that for each $K \in \mathcal{K}(X)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $L \in \mathcal{K}(X)$ with $\widetilde{d}_{\mathcal{H}}(K, L) < \delta$, then $d_{\mathcal{H}}(K, L) < \varepsilon$.
Suppose that this is false. Then, there exist $K \in \mathcal{K}(X)$ and $\varepsilon > 0$ such that for each $n \in \mathbb{N}_{>0}$, there exists $L_n \in \mathcal{K}(X)$ with $\widetilde{d}_{\mathcal{H}}(L_n, K) < 1/n$ (i.e. $L_n \in \widetilde{U}_{1/n}(K)$ and $K \subset \widetilde{U}_{1/n}(L_n)$ by lemma 4.3.2) and $d_{\mathcal{H}}(L_n, K) \geq \varepsilon$ (i.e. $L_n \not\subset U_\varepsilon(K)$ or $K \not\subset U_\varepsilon(L_n)$ by lemma 4.3.2). Since $U_\varepsilon(K)$ is an open neighbourhood of K, there exists n_0 such that $\widetilde{U}_{1/n}(K) \subset U_\varepsilon(K)$ for each $n \geq n_0$. Hence, for each $n \geq n_0$, $L_n \subset U_\varepsilon(K)$, and thus $K \not\subset U_\varepsilon(L_n)$. This proves that for each $n \geq n_0$, there exists $x_n \in K$ such that $d(x_n, L_n) \geq \varepsilon$.
Since $x_n \in K$, we can choose a subsequence (x_{n_k}) with $n_k \geq k$ such that $x_{n_k} \to x \in K$ and for each k, $d(x_{n_k}, L_{n_k}) \geq \varepsilon$ and $\widetilde{d}(x_{n_k}, L_{n_k}) < 1/n_k \leq 1/k$. Hence there exists $y_k \in L_{n_k}$ such that $\widetilde{d}(x_{n_k}, y_k) < 1/k$. Since $x_{n_k} \to x$, we obtain $y_k \to x$. Thus

$$d(x_{n_k}, L_k) \leq d(x_{n_k}, y_k) \leq \underbrace{d(x_{n_k}, x)}_{\to 0} + \underbrace{d(x, y_k)}_{\to 0} \to 0,$$

which is a contradiction with the fact that $d(x_{n_k}, L_k) \geq \varepsilon$ for each k. □

4.3.4 Remark It is possible to define the Hausdorff metric on the set of closed subsets of X (see [Kel55]). But, in this case, the previous theorem is false as it is shown by problem D in [Kel55, Chapter 4].

4.3.5 Proposition *Let (X, d) be a locally compact metric space. If R is an open equivalence relation on X such that each class $R(x)$ is compact, then the following conditions are equivalent:*

(a) *R is proper (i.e. $R(K)$ is compact for each K compact)*

(b) *X/R is Hausdorff*

(c) *For each $a \in X$, $R(a)$ has a fundamental system of open saturated neighbourhoods.*

Proof To prove the equivalence (b)⇔(c), we have to use a classical topological argumentation, which we can find in [Hol78, §4]. The equivalence (a)⇔(c) is proved in [KK83, Proposition 33B.4]. □

4.3.6 Lemma *Let (X, d) be a locally compact metric space and let R be an equivalence relation on X such that each class $R(x)$ is compact. If the canonical mapping $\varphi : X \longrightarrow \mathcal{K}(X)$ given by $\varphi(x) = R(x)$ is continuous then R is an open equivalence relation.*

Proof We have to show that if $x_n \to a$ and if $b \in R(a)$, then there exists a sequence (y_n) such that $y_n \in R(x_n)$ and $y_n \to b$ (see lemma 1.1.2).
Let $N > 0$. Since $x_n \to a$ and φ is continuous, $\varphi(x_n) \to \varphi(a)$. Hence, there exists n_N such that $d_{\mathcal{H}}(R(x_n), R(a)) < 1/N$ for each $n \geq n_N$. Hence, by lemma 4.3.2, $R(a) \subset U_{1/N}(R(x_n))$. Since $b \in R(a)$, for each $n \geq n_N$, there exists $y_n^{(N)} \in R(x_n)$ such that $d(b, y_n^{(N)}) < 1/N$. We can suppose in the construction that $n_1 < n_2 < \cdots$.
Define

$$y_n := \begin{cases} x_n & \text{if } n < n_1 \\ y_n^{(N)} & \text{if } n_N \leq n < n_{N+1}. \end{cases}$$

By definition, $y_n \in R(x_n)$ for each n. Furthermore, if $n_N \leq n < n_{N+1}$, then $d(b, y_n) = d(b, y_n^{(N)}) < 1/N$. Hence $y_n \to b$. □

4.3.7 Proposition *Let (X, d) be a locally compact metric space. If R is an equivalence relation on X such that each class $R(x)$ is compact, then the following conditions are equivalent:*

(a) *The canonical mapping $\varphi : X \longrightarrow \mathcal{K}(X)$ given by $\varphi(x) = R(x)$ is continuous*

(b) *The canonical mapping $\overline{\varphi} : X/R \longrightarrow \mathcal{K}(X)$ given by $\varphi([x]) = R(x)$ is continuous.*

(c) *The canonical mapping $\overline{\varphi} : X/R \longrightarrow \mathcal{K}(X)$ given by $\varphi([x]) = R(x)$ is a homeomorphism onto its image.*

Proof Denote by $\pi : X \longrightarrow X/R$ the canonical projection.
"(a)⇒(b)" By lemma 4.3.6, π is an open mapping. Hence $\overline{\varphi}$ is continuous.
"(b)⇒(c)" Since $\overline{\varphi}$ is injective, it is bijective onto its image. We have to show that $(\overline{\varphi})^{-1}$ is continuous, i.e. for each a sequence (x_k) in X and $a \in X$ such that $R(x_k) \to R(a)$, then $\pi(x_k) \to \pi(a)$.
Let $V \Subset X/R$ be an open neighbourhood of $\pi(a)$. Since $\pi^{-1}(V)$ is open in X, there exists $\varepsilon > 0$ such that $U_\varepsilon(R(a)) \subset \pi^{-1}(V)$. The set $B_\varepsilon^{\mathcal{H}}(R(a))$ is an open neighbourhood of $R(a)$ in $\mathcal{K}(X)$. Hence, there exists n_0 such that $R(x_n) \in B_\varepsilon^{\mathcal{H}}(R(a))$ for each $n \geq n_0$. Thus, by lemma 4.3.2, $R(x_n) \subset U_\varepsilon(R(a)) \subset \pi^{-1}(V)$. Hence $\pi(x_n) \in V$ for each $n \geq n_0$. This proves that $\pi(x_n) \to \pi(a)$.
"(c)⇒(a)" Since $\varphi = \overline{\varphi} \circ \pi$, φ is continuous. □

The following theorem explains when the conditions of proposition 4.3.5 are equivalent to the conditions of proposition 4.3.7.

4.3.8 Theorem *Let (X, d) be a locally compact metric space. If R is an equivalence relation on X such that each class $R(x)$ is compact, then the following conditions are equivalent:*

(a) *R is open and proper*

(b) *The canonical mapping $\varphi : X \longrightarrow \mathcal{K}(X)$ given by $\varphi(x) = R(x)$ is continuous.*

Hence, by proposition 4.3.7, X/R with the quotient-topology is a subspace of $\mathcal{K}(X)$ in a natural way.

Proof "(b)⇒(a)" By proposition 4.3.7, $\overline{\varphi}$ is a homeomorphism on its image. Since $\overline{\varphi}(X/R)$ is Hausdorff, X/R is Hausdorff. The openness of R is proved by lemma 4.3.6.
"(a)⇒(b)" We have to see that for each $a \in X$ and for each $\varepsilon > 0$, there exists an open neighbourhood U of $R(a)$ in X such that if $x \in U$, then $R(x) \in B_\varepsilon^{\mathcal{H}}(R(a))$.
Choose $a \in X$ and $\varepsilon > 0$ and suppose that it is false. Hence for each open neighbourhood $U \Subset X$ of $R(a)$, there exists $x \in U$ such that $d_{\mathcal{H}}(R(x), R(a)) \geq \varepsilon$.
Since X is locally compact, there exists $n_0 > 1/\varepsilon$ such that $U_{1/n}(R(a))$ is relatively compact in X for each $n \geq n_0$. Since R is proper, there exists an open saturated neighbourhood $U_n \Subset X$ of $R(a)$ such that $U_n \subset U_{1/n}(R(a))$. By assumption, there exists $x_n \in U_n$ such that $d_{\mathcal{H}}(R(x_n), R(a)) \geq \varepsilon$. Hence, by lemma 4.3.2, $R(x_n) \not\subset U_\varepsilon(R(a))$ or $R(a) \not\subset U_\varepsilon(R(x_n))$. But, since U_n is saturated, for each $n \geq n_0$,

$$R(x_n) \subset U_n \subset U_{1/n}(R(a)) \subset U_\varepsilon(R(a)).$$

Thus, for each $n \geq n_0$, $R(a) \not\subset U_\varepsilon(R(x_n))$, i.e. there exist $y_n \in R(a)$ such that $d(y_n, R(x_n)) \geq \varepsilon$. Since, for each $n \geq n_0$, $y_n \in \overline{U_{1/n_0}(R(a))}$, which is compact, and $x_n \in U_n \subset \overline{U_{1/n_0}(R(a))}$, we can suppose that $y_n \to y \in R(a)$ and $x_n \to x \in R(a)$. Since R is open, there exist $z_n \in R(x_n)$ such that $z_n \to y$ (compare lemma 1.1.2). Hence

$$d(y_n, R(x_n)) \leq d(y_n, z_n) \leq d(y_n, y) + d(y, z_n) \to 0,$$

which is a contradiction with the fact that $d(y_n, R(x_n)) \geq \varepsilon$ for each $n \geq n_0$. □

4.3.9 Example Let $\sigma : \widetilde{\mathbb{C}^n} \longrightarrow \mathbb{C}^n$ be the blowing-up mapping of $\mathbb{C}^n$ at 0 (see for example [KK83, §32B] or [BK81, §8.4]). Since σ is not open, the equivalence relation R_σ on $\widetilde{\mathbb{C}^n}$ is not open. The fibres of R_σ are given by

$$R_\sigma(x) = \begin{cases} x & \text{if } x \notin \sigma^{-1}(0) \\ \sigma^{-1}(0) \cong \mathbb{P}_{n-1} & \text{if } x \in \sigma^{-1}(0). \end{cases}$$

But, R_σ is proper. The canonical mapping $\varphi : \widetilde{\mathbb{C}^n} \longrightarrow \mathcal{K}(\widetilde{\mathbb{C}^n})$ given by $\varphi(x) = R_\sigma(x)$ is not continuous: if $x_k \to x \in \sigma^{-1}(0)$, with $x_k \notin \sigma^{-1}(0)$, then $\varphi(x_k) \to \{x\} \neq \varphi(x) \cong \mathbb{P}_{n-1}$.

4.3.10 Example Let $\mathbb{F}$ be a compact regular holomorphic foliation on a complex manifold X. The equivalence relation $R^{\mathbb{F}}$ is open. If $\mathbb{F}$ is not stable, then $X/\mathbb{F}$ is not Haudorff, and hence $R^{\mathbb{F}}$ is not proper by proposition 4.3.5. This proves that the canonical mapping $\overline{\varphi} : X/\mathbb{F} \longrightarrow \mathcal{K}(X)$ is not continuous. Inversely, if $\mathbb{F}$ is stable, then $\overline{\varphi}$ is continuous.

4.4 The set $\widetilde{Z}_d(X)$ of analytic subsets of X

In this subsection, we define the set $\widetilde{Z}_d(X)$ and impose a topology on it. This topology is a quotient-topology of $Z_d(X)$.

In this subsection, X denotes a paracompact complex manifold of dimension n.

Denote

$$\widetilde{Z}_d(X) := \{A \subset X \mid A \text{ is analytic in } X \text{ and is of pure dimension } d\}.$$

Let R_O be the equivalence relation on $Z_d(X)$ given by

$$Z \underset{R_O}{\sim} Z' \;:\Longleftrightarrow\; |Z| = |Z'|.$$

Then $\widetilde{Z}_d(X) = Z_d(X)/R_O$. The topology imposed on $\widetilde{Z}_d(X)$ is the quotient-topology. We call it the ***Barlet-topology***. Similarly, $\widetilde{C}_d(X) = C_d(X)/R_O$.

4.4.1 Lemma *If $\mathcal{S}$ is a d-dimensional scale, then the sets $B_{\mathcal{S}}(0)$ and $\bigcup_{k\geq 1} B_{\mathcal{S}}(k)$ are open in $Z_d(X)$ and R_O-saturated.*

Proof These sets are open. We have to show that they are saturated.
If $Z \in B_{\mathcal{S}}(0)$, then $|Z| \cap \overline{|\mathcal{S}|} = \emptyset$. Hence if $Z' \in R_O(Z)$, then $|Z'| \cap \overline{|\mathcal{S}|} = \emptyset$ and hence $Z' \in B_{\mathcal{S}}(0)$.
If $Z \in \bigcup_{k\geq 1} B_{\mathcal{S}}(k)$ and $Z' \in R_O(Z)$, then $\mathcal{S}$ is adapted to Z' and $\deg_{\mathcal{S}}(Z') \neq 0$. Hence $Z' \in \bigcup_{k\geq 1} B_{\mathcal{S}}(k)$.

4.4.2 Remark In general, R_O is not an open equivalence relation. There may exist scales $\mathcal{S}$ and natural numbers $k \geq 1$ such that $R_O(B_{\mathcal{S}}(k))$ is not open, as it is shown by example 4.4.3

4.4.3 Example Let $X := \mathbb{C}^2$. Let $\mathcal{S} := (\mathrm{Id}, D, D)$ be a 1-dimensional scale on X, where $D := \{z \in \mathbb{C} \mid |z| < 1\}$. We prove that $R_O(B_{\mathcal{S}}(1))$ is not open in $Z_1(X)$. For that, let $Z := [\{z_1 = 0\}]$ and $Z' := 2 \cdot [\{z_1 = 0\}]$. The cycles Z and Z' are elements of $Z_1(X)$. Furthermore, $Z \in B_{\mathcal{S}}(1)$ and $Z' \in R_O(B_{\mathcal{S}}(1)) \setminus B_{\mathcal{S}}(1)$. It suffices to prove that there exists a sequence (Z'_k) in $Z_1(X)$ such that $Z'_k \to Z'$, but $Z'_k \notin R_O(B_{\mathcal{S}}(1))$. Let $Z'_k := [\{1/k\} \times \mathbb{C}] + [\{-1/k\} \times \mathbb{C}]$. One easily sees that $Z'_k \to Z'$. But, $\deg_{\mathcal{S}}(Z'_k) = 2$ for each $k \geq 2$ and Z'_k has multiplicities 1. Hence, there does not exist $Z_k \in B_{\mathcal{S}}(1)$ such that $|Z'_k| = |Z_k|$, which completes the proof.

4.4.4 Proposition $\widetilde{Z}_d(X)$ is Hausdorff.

Proof Let $Z, Z' \in Z_d(X)$ such that $|Z| \neq |Z'|$. Then there exists a scale $\mathcal{S}$ adapted to Z and Z' such that $|Z| \cap |\mathcal{S}| = \emptyset$ and $|Z'| \cap |\mathcal{S}| \neq \emptyset$. Hence $Z \in B_{\mathcal{S}}(0)$ and $Z' \in \bigcup_{k\geq 1} B_{\mathcal{S}}(k)$ (these sets are open and R_O-saturated by lemma 4.4.1) and $B_{\mathcal{S}}(0) \cap (\bigcup_{k\geq 1} B_{\mathcal{S}}(k)) = \emptyset$. $\square$

5 Stability and multiplicities of leaves of regular foliations

In this subsection, we apply results of section 4 to define, for a regular foliations $\mathbb{F}$, the stability and the topological multiplicity of a $\mathbb{F}$-leaf and the good set $G(\mathbb{F})$ of $\mathbb{F}$. At the end, we define also the analytical multiplicity of a $\mathbb{F}$-leaf, the mapping $\zeta_{\mathbb{F}}: X \longrightarrow Z_d(X)$ and the set $C(\mathbb{F})$.

5.1 The notion of stability for non-compact regular foliations

A compact leaf of a regular foliation is called ***stable*** if it has a fundamental system of open saturated neighbourhoods (see for example [Hol78]). This definition is not usable for non-compact leaves. In this subsection we propose two definitions of stability for non-compact leaves.

5.1.1 Definition A leaf L of a regular foliation $\mathbb{F}$ is called ***stable of type one*** (or ***1-stable***) if $L \in \mathcal{H}_1(X/\mathbb{F})$
A leaf L of a regular foliation $\mathbb{F}$ is called ***stable of type two*** (or ***2-stable***) if $L \in \mathcal{H}_2(X/\mathbb{F})$.

We use the following notations:

$$X^1_{\mathrm{st}}(\mathbb{F}) := \{x \in X \mid L_x \text{ is 1-stable}\} = \pi^{-1}(\mathcal{H}_1(X/\mathbb{F}))$$

and

$$X^2_{\mathrm{st}}(\mathbb{F}) := \{x \in X \mid L_x \text{ is 2-stable}\} = \pi^{-1}(\mathcal{H}_2(X/\mathbb{F})).$$

We write X^1_{st} and X^2_{st} if it is not necessary to precise the foliation.

5.1.2 Proposition *These two sets have the following properties*

(a) X^1_{st} *is open in* X
(b) $X^1_{\mathrm{st}} \subset X^2_{\mathrm{st}}$
(c) X^1_{st} *is* $\mathbb{F}$*-saturated*
(d) X^2_{st} *is* $\mathbb{F}$*-saturated*
(e) $X^1_{\mathrm{st}}/\mathbb{F}$ *is Hausdorff*
(f) $X^2_{\mathrm{st}}/\mathbb{F}$ *is Hausdorff*

Proof (a) and (b) follow from proposition 4.1.2. (c) and (d) follow from the definition of X^1_{st} and X^2_{st}. (e) and (f) follow from proposition 4.1.5, because $X^j_{\mathrm{st}}/\mathbb{F} = \mathcal{H}_j(X/\mathbb{F})$. $\square$

To see that these definitions are reasonable, we need to see that they coincide with the definition of stable leaves in the compact case.

5.1.3 Proposition *Let $\mathbb{F}$ be a compact regular foliation. Then*

$$\{x \in X \mid L_x \text{ is stable}\} = X^1_{\mathrm{st}} = X^2_{\mathrm{st}}.$$

Proof Define $A := \{x \in X \mid L_x \text{ is stable}\}$.
$A \subset X^1_{\mathrm{st}}$: Let $x \in A$, i.e. L_x is stable. By [Hol78, Lemma 2.5] and [Hol78, Proposition 4.2], $A \Subset X$. Hence there exists an open $\mathbb{F}$-saturated neighbourhood V of x such that each leaf L in V is stable. Let $V' \Subset V$ be an open neighbourhood of L_x such that $\overline{V'} \subset V$. Since L_x is stable, there exists an open $\mathbb{F}$-saturated neighbourhood $U \Subset V'$ of L_x. Hence $\overline{U} \subset V$. Thus $\pi(\overline{U}) = \overline{\pi(U)}$ because π is open, and $\overline{\pi(U)}$ is Hausdorff, because $V/\mathbb{F}$ is Hausdorff and $\pi(\overline{U}) \subset V/\mathbb{F}$. Hence $x \in X^1_{\mathrm{st}}$.
$X^1_{\mathrm{st}} \subset X^2_{\mathrm{st}}$ follows from proposition 5.1.2(b).
$X^2_{\mathrm{st}} \subset A$: Let $x \in X^2_{\mathrm{st}}$ and let $W \Subset X$ be an open neighbourhood of L_x. We have to see that there exists an open $\mathbb{F}$-saturated neighbourhood $U \subset W$ of L_x. Let $V \Subset W$ be an open neighbourhood of L_x such that $\overline{V} \subset W$ is compact in X (this is possible because L_x is compact). Thus $\pi(\partial V)$ is quasicompact in $X/\mathbb{F}$. By construction, $L_x \notin \pi(\partial V)$. By lemma 4.1.11, there exists an open neighbourhood $\widetilde{U} \Subset X/\mathbb{F}$ of L_x such that $\widetilde{U} \cap \pi(\partial V) = \emptyset$. By the connectedness of leaves of $\mathbb{F}$, the open set $U := V \cap \pi^{-1}(\widetilde{U})$ is $\mathbb{F}$-saturated. □

5.1.4 Definition A regular foliation is called ***stable*** if $X/\mathbb{F}$ is Hausdorff

5.1.5 Lemma *The following conditions are equivalent:*

(a) $\mathbb{F}$ *is stable*
(b) $X^1_{\mathrm{st}} = X$
(c) $X^2_{\mathrm{st}} = X$

Proof This is a consequence of theorem 4.1.6 □

5.2 The good set of a regular foliation and the topological multiplicity of a leaf

In this subsection, we define the good set $G(\mathbb{F})$ of a regular foliation $\mathbb{F}$. In a second part, we define the notion of topological multiplicity, which is linked to the notion of R-multiplicity.

If (U, p) is a local $\mathbb{F}$-foliation, we consider the equivalence relation R_p on $V := p(U)$ defined by:

$$v \underset{R_p}{\sim} v' \quad :\Longleftrightarrow \quad p^{-1}(v) \text{ and } p^{-1}(v') \text{ belong to the same leaf of } \mathbb{F}.$$

5.2.1 Lemma *The equivalence relation R_p satisfies the following properties:*

(a) *R_p is open*

(b) *for each $x \in U$, the equivalence class $R_p(p(x))$ is either discrete in V or has no isolated points at all.*

(c) *$R_p \subset V \times V$ is weakly-analytic*

Furthermore, $R_p(p(x))$ is discrete iff L_x is closed in X.

For the proof see [Hol72, Lemma 2.10]. □

The following example gives an example of a foliation, for which all leaves are dense in X.

5.2.2 Example Let $T := \mathbb{C}/G$ be the complex torus, where $G := \mathbb{Z} + i\mathbb{Z}$. Let $X := T \times T$. We consider the foliation $\mathbb{F}$ given by the following action of the group $(\mathbb{C}, +)$ on X:

$$z \cdot ([x_1], [x_2]) := ([x_1 + z], [x_2 + z\pi]), \quad z \in \mathbb{C},\ ([x_1], [x_2]) \in X.$$

Since the set $A := \{a\pi + b \mid a, b \in \mathbb{Z}\} \subsetneq \mathbb{R}$ is dense in $\mathbb{R}$, the leaf $L := \{([z], [z\pi]) \in X \mid z \in \mathbb{C}\}$ of $\mathbb{F}$ is dense in X. By translation, all leaves of $\mathbb{F}$ are dense in X.

5.2.3 Notation If (U, p) is a local $\mathbb{F}$-foliation such that R_p is quasi-finite, then we denote the mapping ν_{R_p} by ν_p, i.e $\nu_p : p(U) \longrightarrow \mathbb{N}$ is given by $\nu_p(v) = \mathrm{Card}(R_p(v))$.

5.2.4 Lemma *If $\mathbb{F}$ is a regular foliation with all leaves closed, then for each $x \in X$ there exists a local $\mathbb{F}$-foliation (U, p) at x such that R_p is quasi-finite.*

Proof Let $x \in X$ and (U', p') be a local $\mathbb{F}$-foliation at x. By lemma 5.2.1, $R_{p'}(v)$ is discrete in V' for all $v \in V'$. We choose another local $\mathbb{F}$-foliation (U, p) at x, such that $\overline{U} \subset U'$ is compact in X and $p = p'|_U$. For each $v \in \overline{V}$ the set $R_{p'}(v) \cap \overline{V}$ is discrete in $\overline{V}$. By the compactness of $\overline{V}$, the set $R_{p'}(v) \cap \overline{V}$ is finite. Hence R_p is quasi-finite. □

5.2.5 Definition Let $\mathbb{F}$ be a regular foliation. The set

$$G(\mathbb{F}) := \{x \in X \mid \text{there exists a local } \mathbb{F}\text{-foliation } (U, p) \text{ at } x \text{ such that } R_p \text{ is quasi-finite and bounded}\}.$$

is called the ***good set*** of $\mathbb{F}$. We write G if it is not necessary to precise the foliation.

5.2.6 Proposition *G has the following properties*

(a) *G is $\mathbb{F}$-saturated*

(b) *G is open in X*

(c) *Each leaf in G is closed*

Proof Ad (a). Let $x \in G$ and $y \in L_x$. Let (U,p) be a local $\mathbb{F}$-foliation at x, such that R_p is quasi-finite and bounded. Let (U',p') be a local $\mathbb{F}$-foliation at y. By lemma 3.1.6, we can restrict V and V' such that there exists a biholomorphic mapping $h\colon V' \longrightarrow V$ with $h(p'(y)) = p(x)$, such that for each $v \in V'$, the points $(p')^{-1}(v)$ and $p^{-1}(h(v))$ belong to the same leaf. Let $v \in V'$. By properties of h, $R_p(h(v)) \supset h(R_{p'}(v))$: if $\widetilde{v} \in h(R_{p'}(v))$, then $h^{-1}(\widetilde{v}) \in R_{p'}(v)$ and finally $p^{-1}(\widetilde{v})$, $(p')^{-1}(h^{-1}(\widetilde{v}))$, $(p')^{-1}(v)$ and $p^{-1}(h(v))$ belong to the same leaf, which implies that $\widetilde{v} \in R_p(h(v))$. Since h is bijective, $R_{p'}$ is quasi-finite and $\operatorname{Card} R_p(h(v)) \geq \operatorname{Card} R_{p'}(v)$. Hence

$$\nu_{p'}(v) \leq \nu_p(h(v)) \leq \max_{w \in V} (\nu_p(w)) < \infty,$$

which proves that $R_{p'}$ is bounded. Thus $y \in G$.
Ad (b). Let $x \in G$. Let (U,p) be a local $\mathbb{F}$-foliation at x, such that R_p is bounded. Let $y \in U$ and (U',p') be a local $\mathbb{F}$-foliation at y, such that $U' \subset U$ and $p' = p|_{U'}$. Thus, for each $u \in U$,

$$R_{p'}(p'(u)) = R_p(p(u)) \cap p(U') \subseteq R_p(p(u))$$

and hence $\nu_{p'}(p'(u)) \leq \nu_p(p(u)) \leq \max_{v \in V} (\nu_p(v)) < \infty$. Thus $R_{p'}$ is bounded and $y \in G$.
Property (c) is a consequence of lemma 5.2.1. □

5.2.7 Lemma *Let $x \in G$ and $x' \in L_x$. If (U,p), resp. (U',p'), is a local $\mathbb{F}$-foliation at x, resp. at x', then $\mu_{R_p}(p(x)) = \mu_{R_{p'}}(p'(x'))$.*

Proof By lemma 3.1.6 we can restrict U and U' such that there exists a biholomorphic mapping $h\colon V \longrightarrow V'$ with $h(p(x)) = p'(x')$ such that for each $v \in V$, the points $p^{-1}(v)$ and $(p')^{-1}(h(v))$ belong to the same leaf. By proposition 4.2.4(b) this restriction does not change the values $\mu_{R_p}(p(x))$ and $\mu_{R_{p'}}(p'(x'))$.
If $v \in V$ then $h(R_p(v)) \subset R_{p'}(h(v))$ (as in proof of proposition 5.2.6). Similarly, $h(R_p(v)) \supset R_{p'}(h(v))$, because h^{-1} has the same properties as h. Hence $h(R_p(v)) = R_{p'}(h(v))$ for each $v \in V$. Since h is bijective, the following equations hold:

$$\begin{aligned}
\mu_{R_p}(p(x)) &= \min_{x \in W \Subset V} \Big(\max_{v \in W} (\operatorname{Card}(R_p(v) \cap W)) \Big) \\
&= \min_{h(x) \in W' \Subset V'} \Big(\max_{v \in h^{-1}(W')} \Big(\operatorname{Card}(h(R_p(v)) \cap W') \Big) \Big) \\
&= \min_{x' \in W' \Subset V'} \Big(\max_{v' \in W'} (\operatorname{Card}(R_{p'}(v') \cap W')) \Big) = \mu_{R_{p'}}(p'(x)).
\end{aligned}$$

This completes the proof. □

5.2.8 Definition For $x \in G$ the ***topological multiplicity*** $\mu_t(L_x)$ of the leaf L_x is given by

$$\mu_t(L_x) := \mu_{R_p}(p(x)),$$

where (U, p) is a local $\mathbb{F}$-foliation at x.

5.2.9 Remark Note that μ_t is well-defined by lemma 5.2.7.

5.2.10 Example Let $\mathbb{F}$ be the foliation on $X := \mathbb{C}^2 \setminus \{0\}$ given by the mapping $f : X \longrightarrow \mathbb{C}$, $f(x) = x_1^2 x_2^3$ (see example 2.3.2). The sets $A := \{x \in X \mid x_2 = 0\}$ and $B := \{x \in X \mid x_1 = 0\}$ are two different leaves of $\mathbb{F}$.
The mapping f is not a submersion. There exists an open neighbourhood $U \Subset X$ of $(1, 0) \in X$ such that $U \cap B = \emptyset$ and the mapping $(z_1, z_2) \mapsto \sqrt[3]{z_1^2}$ is well-defined and holomorphic for each $(z_1, z_2) \in U$. Furthermore

$$f(z_1, z_2) = z_1^2 z_2^3 = \left(\sqrt[3]{z_1^2}\right)^3 z_2^3 = \left(\sqrt[3]{z_1^2} z_2\right)^3.$$

Hence, for each $z \in U$, $f(z) = (h(z))^3$, where $h(z) := \sqrt[3]{z_1^2} z_2$ is holomorphic on U. Since h is a submersion, there exist an open neighbourhood $V \Subset U$ of $(1, 0)$, an open subset W of $\mathbb{C}$ and a biholomorphism $\alpha : V \longrightarrow h(V) \times W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, "h"'] & h(V) \times W \arrow[d, "\mathrm{pr}_1"] \\ & h(V) \end{tikzcd}$$

commutes. Hence (V, h) is a local $\mathbb{F}$-foliation. Two points $x, y \in V$ belong to the same leaf if $(h(x))^3 = (h(y))^3$. Hence, for each $v \in h(V)$, $R_h(v) = \{\widetilde{v} \in h(V) \mid \widetilde{v}^3 = v^3\}$. Hence $\mu_{R_h}(0) = 3$. Thus $\mu_t(A) = \mu_{R_h}(h(1, 0)) = 3$. Similarly $\mu_t(B) = 2$.

5.2.11 Lemma *If $x \in G$, then there exists a local $\mathbb{F}$-foliation (U, p) at x, such that*

$$\mu_t(L_x) = \max_{v \in p(U)} (\nu_p(v)).$$

Proof This is a consequence of proposition 4.2.4(a). □

The next theorem explains that the topological multiplicity of a leaf of a compact foliation is exactly the number of elements of the holonomy group of that leaf.

5.2.12 Theorem *If $\mathbb{F}$ is a compact regular foliation, then $\mu_t(L_x) = \mathrm{Card}\, H(L_x)$ for each $x \in G$, where $H(L_x)$ is the holonomy group of the leaf L_x (in the sense of [Hol78, §3]).*

Proof By [Hol78, Theorem 4.2], $G/\mathbb{F}$ is a Hausdorff space and consequently a complex space. By [Hol72, Lemma 3.3] there exists a local $\mathbb{F}$-foliation (U, p) at x, such that the equivalence relation R_p is open, finite and analytic, i.e. $\pi_p : V \longrightarrow V/R_p$

is an holomorphic finite and open mapping, and hence π_p is an analytic covering. By lemma 5.2.11, we can restrict U such that

$$\mu_t(L_x) = \max_{v \in V} R_p(v) = \text{number of sheets of } \pi_p.$$

By [Hol78, Lemma 3.2] we can again restrict U such that there exists a group $H(V)$ of biholomorphic mappings of V onto itself with $p(x)$ as fixed point such that $(v, h(v)) \in R_p$ for each $v \in V$, $H(L_x) = \{h_{p(x)}, h \in H(V)\}$ and $\operatorname{ord} H(L_x) = \operatorname{ord} H(V)$.
If π_p is unbranched in $v \in V$, then $\pi_p^{-1}(\pi_p(v))$ has $\mu_t(L_x)$ elements and we note $\pi_p^{-1}(\pi_p(v)) =: \{v_1, \dots, v_{\mu_t(L_x)}\}$, where $v = v_1$. Furthermore there exist $\mu_t(L_x)$ pairwise disjointed connected open subsets V_j of V such that, for each $j = 1, \dots \mu_t(L_x)$, V_j is a neighbourhood of v_j, π_p is biholomorphic on V_j and $\pi_p^{-1}(\pi_p(V_j)) = \bigcup_{k=1}^{\mu_t(L_x)} V_k$. For each $j = 1, \dots, \mu_t(L_x)$, there exists a mapping $h \in H(V)$ such that $h(v) = v_j$. Hence $\operatorname{Card} H(L_x) \geq \mu_t(L_x)$. We claim that

$$\text{if } h \in H(V) \text{ with } h(p(y_j)) = p(y_k) \text{ for one } j \text{ and one } k, \text{ then } h(V_j) = V_k. \qquad (*)$$

If there exist two mappings $h, h' \in H(V)$ and on j such that $h(v) = h'(v) = v_j$, then $h(V_1) = h'(V_1) = V_j$ by $(*)$. Thus $h = h'$ by identity theorem. Hence $\operatorname{Card} H(L_x) = \mu_t(L_x)$.
We prove $(*)$. Let $v \in h(V_j)$, i.e. $v = h(v')$, $v' \in V_j$. Then $(v', v) \in R_p$ and hence $v \in \pi_p^{-1}(\pi_p(v)) \subset \bigcup_{l=1}^{\mu_t(L_x)} V_l$. We conclude that $h(V_j) \subset \bigcup_{l=1}^{\mu_t(L_x)} V_l$. Since V_j is connected, $h(V_j)$ is also connected. Hence, there exists l_0 such that $h(V_j) \subset V_{l_0}$. Since $h(p(y_j)) = p(y_k)$, $h(V_j) \subset V_k$. We can argue similarly to prove that $h^{-1}(V_k) \subset V_j$. Hence $h(V_j) = V_k$. □

5.2.13 Definition Let $\mathbb{F}$ be a regular foliation. The set

$$X_{\mathrm{tr}}(\mathbb{F}) := \{x \in G \mid \mu_t(L_x) = 1\}$$

is called the ***trivial locus*** of $\mathbb{F}$. We write X_{tr} if it is not necessary to precise the foliation. A leaf L is called ***trivial*** if $L \subset X_{\mathrm{tr}}$.

5.2.14 Proposition *The following equation holds:*

$$X_{\mathrm{tr}} = \{x \in X \mid \textit{there exists a local } \mathbb{F}\textit{-foliation } (U, p) \textit{ at } x \textit{ such that } R_p = \Delta_{p(U)}\}.$$

Furthermore, if (U, p) is a local $\mathbb{F}$-foliation, then

$$X_{\mathrm{tr}} \cap U = p^{-1}(V_{\mathrm{tr}}^{R_p}).$$

□

5.2.15 Example Let $X := \mathbb{C}^2 \setminus \{0\}$. We consider the foliation $\mathbb{F}$ on X given by the mapping $(z_1, z_2) \mapsto z_1 z_2$. In this case, $X_{\mathrm{tr}} = X$.

5.2.16 Proposition *X_{tr} has the following properties*

(a) *X_{tr} is $\mathbb{F}$-saturated*

(b) *X_{tr} is open in X*

Proof Property (a) is a consequence of lemma 5.2.7. Property (b) follows from proposition 5.2.14 and theorem 4.2.7. $\square$

5.2.17 Theorem *If $\mathbb{F}$ is a regular foliation with all leaves closed, then X_{tr} and G are dense in X.*

Proof Let $U' \Subset X$ be an open in X. To prove that $X \setminus X_{\mathrm{tr}}$ is nowhere dense in X, we have to find an open subset $\widetilde{U}$ of U' such that $\widetilde{U} \subset X_{\mathrm{tr}}$. Let $x \in U'$ and let (U, p) be a local $\mathbb{F}$-foliation at x, with $U \Subset U'$, such that the equivalence relation R_p on $V := p(U)$ is quasi-finite (this is possible by lemma 5.2.4). By assumptions on $\mathbb{F}$ and by lemma 5.2.1, R_p is an open, weakly-analytic and quasi-finite equivalence relation on V. By theorem 4.2.7 the set $V_{\mathrm{tr}}^{R_p}$ is open and dense in V. Let $\widetilde{U} \Subset p^{-1}(V_{\mathrm{tr}}^{R_p})$. Then $\widetilde{U} \subset X_{\mathrm{tr}}$ by proposition 5.2.14.
Since $X_{\mathrm{tr}} \subset G$, G is dense in X. $\square$

5.2.18 Remark If $\mathbb{F}$ has non-closed leaves, then G is not dense in X in general. This is for example the case with example 5.2.2.

5.3 The analytic multiplicity of a leaf

In this subsection, we define a new multiplicity on leaves, called the analytic multiplicity.

In this subsection, $\mathbb{F}$ denotes always a regular foliation.

If $X/\mathbb{F}$ is a complex space, then $X/\mathbb{F}$ is normal. Since π is open, $(X/\mathbb{F})_{\mathrm{gen}(\pi)} = X/\mathbb{F}$ by remark 2.3.4. By remark 2.3.6, $|Z_\pi(\pi(x))| = L_x$ for each $x \in X$. By that we can define:

5.3.1 Definition If $X/\mathbb{F}$ is a complex space and $x \in X$, the ***analytic multiplicity*** $\mu_a(L_x)$ of the leaf L_x of $\mathbb{F}$ through x is the coefficient of the cycle $Z_\pi(\pi(x))$, i.e. $Z_\pi(\pi(x)) = \mu_a(L_x)[L_x]$.

5.3.2 Calculation of $\mu_a(L)$ If $X/\mathbb{F}$ is a complex space and $L \in X/\mathbb{F}$, we can calculate the number $\mu_a(L)$ in the following way (see section 2.3): we choose a local

$\mathbb{F}$-foliation (U, p) such that $\emptyset \neq L \cap U$ is connected. Let $V' := \pi(U) \Subset X/\mathbb{F}$. We adopt the notations of definition 3.1.5. We consider the mapping

$$\begin{array}{rcccl} (\pi, q) & : & U & \rightarrow & V' \times W \\ & & x & \mapsto & (\pi(x), q(x)). \end{array}$$

We choose a point $y \in L \cap U$. Then $(\pi, q)^{-1}((\pi, q)(y)) = \{y\}$. Hence, by [GR84, 3.1.2] we can restrict U, $y \in U$, such that the mapping (π, q) is an analytic covering. The analytic multiplicity $\mu_a(L)$ is the number of sheets of the covering (π, q).

5.3.3 Example Let $\mathbb{F}$ be the foliation on $X := \mathbb{C}^2 \setminus \{x \in \mathbb{C}^2 \mid x_1 = 0\}$ given by the mapping of example 2.3.2, i.e. by $f : X \longrightarrow \mathbb{C}$ with $f(x) = x_1^2 x_2^3$. The quotient $X/\mathbb{F}$ is a complex space and is isomorphic to $\mathbb{C}$. Furthermore, the canonical projection is f. Thus, by doing same calculation as in example 2.3.2, one obtains $\mu_a(L_{(2,0)}) = 3$.

5.3.4 Theorem *If $\mathbb{F}$ is a regular foliation such that $X/\mathbb{F}$ is a complex space, then $X = G$ and $\mu_t(L) = \mu_a(L)$ for each $L \in X/\mathbb{F}$.*

Proof We have to show first that for each $x \in X$ there exists a local $\mathbb{F}$-foliation (U, p) at x such that R_p is bounded. Let $x \in X$. By [Hol72, Lemma 3.3], there exists a local $\mathbb{F}$-foliation (U, p) at x such that R_p is open, finite and analytic. Since R_p is proper, it is also bounded. This proves that $X = G$.
The equality $X = G$ says that $\mu_t(L)$ exists for each $L \in X/\mathbb{F}$. Let $L \in X/\mathbb{F}$. By calculation 5.3.2 we can find a local $\mathbb{F}$-foliation (U, p) such that $L \cap U$ is connected and the mapping $(\pi, q) : U \longrightarrow \pi(U) \times W$, is an analytic covering. The number $\mu_a(L)$ is exactly the number of sheets of (π, q). Let $(L', w) \in (\pi(U), W)$ be a unbranched point of (π, q). Thus

$$\mu_a(L) = \operatorname{Card}((\pi, q)^{-1}(L', w)) = \operatorname{Card}(L' \cap q^{-1}(w)) = \max_{v \in V}(\operatorname{Card} R_p(v)) = \mu_t(L),$$

which concludes the proof. $\square$

5.3.5 Proposition *If $X/\mathbb{F}$ is a complex space, then the set $X \setminus X_{\mathrm{tr}}$ is analytically thin in X.*

Proof The set $\operatorname{Sg} \pi = \pi^{-1}(\operatorname{Sing} X/\mathbb{F}) \cup \operatorname{Sing} \pi$ is thin analytic in X. Furthermore, $\pi|_{X \setminus \operatorname{Sg} \pi}$ is a holomorphic submersion (compare lemma 1.2.7). Hence by lemma 5.3.6, $X \setminus \operatorname{Sg} \pi \subset X_{\mathrm{tr}}$, and thus $X \setminus X_{\mathrm{tr}} \subset \operatorname{Sg} \pi$. This implies that $X \setminus X_{\mathrm{tr}}$ is analytically thin in X. $\square$

5.3.6 Lemma *If $X/\mathbb{F}$ is a complex space, then*

$$x \in X_{\mathrm{tr}} \quad \Longleftrightarrow \quad \pi \textit{ is a submersion at } x.$$

Proof "$\Rightarrow$" Let $x_0 \in X_{\mathrm{tr}}$ and let (U, p) be a local $\mathbb{F}$-foliation at x_0 such that $R_p = \Delta_{p(V)}$. Then

$$\pi(U) \cong p(U)/R_p = p(U).$$

Thus $\pi(U)$ is a manifold and π is given on U by a projection. Hence π is a submersion at x_0.
"$\Leftarrow$" Suppose that π is a submersion at $x_0 \in X$. Then there exists an open neighbourhood $U \Subset X$ of x_0 such that $\pi(U)$ has no singularities, an open neighbourhood $W \Subset \mathbb{C}^d$ and a biholomorphism $\alpha : U \longrightarrow \pi(U) \times W$ such that the following diagram commutes:

$$\begin{tikzcd} & \pi(U) \times W \arrow[d, "\mathrm{pr}_1"] \\ U \arrow[ur, "\alpha"] \arrow[r, "\pi"] & \pi(U) \end{tikzcd} .$$

Thus $(U, \pi|_U)$ is a local $\mathbb{F}$-foliation such that $U \subset X_{\mathrm{tr}}$. $\square$

The following definition gives us a canonical method to obtain a scale from a local $\mathbb{F}$-foliation.

5.3.7 Definition Let (U, p) be a local $\mathbb{F}$-foliation. By definition 3.1.5, there exists another local $\mathbb{F}$-foliation $(\widehat{U}, \widehat{p})$ such that (U, p) is a shrinking of $(\widehat{U}, \widehat{p})$. The triple $\mathcal{S} := (\widehat{\alpha} : \widehat{U} \longrightarrow \widehat{V} \times \widehat{W}, V, W)$ is a d-dimensional scale of X. It is called a ***scale associated to the local*** $\mathbb{F}$***-foliation*** $(\boldsymbol{U}, \boldsymbol{p})$.

5.4 The mapping $\zeta_{\mathbb{F}}$ and the set $C(\mathbb{F})$

In this subsection, we define the mapping $\zeta_{\mathbb{F}}$. The continuity of this mapping is in relation with the fact that the quotient space is Hausdorff (see subsection 6.1). In addition we define the set $C(\mathbb{F})$ which is important in the last part of the thesis.

5.4.1 Definitions If $\mathbb{F}$ is a regular foliation we define the mapping $\zeta_{\mathbb{F}} : G \longrightarrow Z_d(X)$ by

$$\zeta_{\mathbb{F}}(x) := \mu_t(L_x)[L_x].$$

In addition, we define the open subset

$$C(\mathbb{F}) := \text{Interior of } \{x \in G \mid \zeta_{\mathbb{F}} \text{ is continuous at } x\}$$

of G. We write C if it is not necessary to precise the foliation.

5.4.2 Remark In general $C \subsetneq G$ (see example 5.4.3). Furthermore, if $U \Subset X$ is $\mathbb{F}$-saturated, then in general $C(\mathbb{F}|_U) \neq C(\mathbb{F}) \cap U$. If $\mathbb{F}$ is a compact foliation, then $C(\mathbb{F}) = G$ (see proposition 6.1.4).

5.4.3 Example In example 5.2.15, $G = X$ and the mapping $\zeta_{\mathbb{F}}$ is not continuous in points $(0, t)$ as in points $(t, 0)$. For the proof, let $(x_n)_{n \in \mathbb{N}} \subset (\mathbb{C}^* \times \mathbb{C}^*)$ be a sequence such that $x_n \to (0, t)$. Then

$$\lim \zeta_{\mathbb{F}}(x_n) = [\{z_1 z_2 = 0\}] \neq [\{z_1 = 0\}] = \zeta_{\mathbb{F}}(0, t).$$

Hence $C(\mathbb{F}) = X \smallsetminus (\{z_1 = 0\} \cup \{z_2 = 0\})$. Furthermore, if $U = X \smallsetminus \{z_2 = 0\}$, then

$$U = C(\mathbb{F}|_U) \neq C(\mathbb{F}) \cap U.$$

5.4.4 Proposition *The set C has the following properties:*

(a) *C is open in X*

(b) *C is $\mathbb{F}$-saturated*

Proof It suffices to prove (b). Let $x \in C$ and $y \in L_x$. There exists an open neighbourhood $U \Subset G$ of x such that $\zeta_{\mathbb{F}}$ is continuous on U. Since $\zeta_{\mathbb{F}}$ is $\mathbb{F}$-invariant, $\zeta_{\mathbb{F}}$ is continuous on $R^{\mathbb{F}}(U)$. Hence $y \in C$. ☐

The following theorem shows that set of the points where $\zeta_{\mathbb{F}}$ is continuous is strongly connected to the points of X^1_{st} and X^2_{st}. The proof of this theorem uses theorem 6.1.1.

5.4.5 Theorem *If $\mathbb{F}$ is a regular holomorphic foliation, then the following conditions hold:*

(a) $C = X^1_{\mathrm{st}}$

(b) $\{x \in G \mid \zeta_{\mathbb{F}}$ *is continuous in* $x\} \ \subset X^2_{\mathrm{st}}$.

Proof Ad (a). By proposition 5.1.2 (e), the quotient $X^1_{\mathrm{st}}/\mathbb{F}$ is Hausdorff. By theorem 6.1.1, $\zeta_{\mathbb{F}}|_{X^1_{\mathrm{st}}}$ is continuous. Hence $X^1_{\mathrm{st}} \subset C$, because X^1_{st} is open in X. Let $x \in C$. There exists an open neighbourhood $U \Subset X$ of x such that $\overline{U} \subset C$. Hence $\pi(\overline{U}) \subset C/\mathbb{F}$. Thus $\pi(\overline{U})$ is Hausdorff by claim 5.4.6. Since π is open, $\pi(\overline{U}) = \overline{\pi(U)}$. Thus $x \in X^1_{\mathrm{st}}$. This concludes the proof of $C = X^1_{\mathrm{st}}$.
Ad (b). Let $x_1 \in G$ such that $x_1 \notin X^2_{\mathrm{st}}$. We have to show that $\zeta_{\mathbb{F}}$ is not continuous in x_1.
Denote $L_1 := L_{x_1}$. By the choice of x_1, there exists a leaf $L_2 \neq L_1$ such that for each $\mathbb{F}$-saturated open neighbourhoods U of L_1 and U' of L_2, $U \cap U' \neq \emptyset$. Choose $x_2 \in L_2$. Since X is paracompact, it is regular (in a topological sense, see for example [Eng89]). Hence, since L_1 is closed in X, there exist an open neighbourhood U of L_1 in X and an open neighbourhood U' of x_2 in X such that $U \cap U' = \emptyset$. Hence we can choose a local $\mathbb{F}$-foliation (U_1, p_1) at x_1 and a local $\mathbb{F}$-foliation (U_2, p_2) at x_2 such that $U_2 \cap L_1 = \emptyset$ and $U_1 \cap U_2 = \emptyset$.
For $j = 1, 2$, we consider the scale $\mathcal{S}_j$ associated to the local $\mathbb{F}$-foliation (U_j, p_j). If it is necessary, we restrict U_j such that $\mathcal{S}_j$ is adapted to $\zeta_{\mathbb{F}}(x_1)$. Let

$$A := B_{\mathcal{S}_1}(\deg_{\mathcal{S}_1}(\zeta_{\mathbb{F}}(x_1))) \ \cap \ B_{\mathcal{S}_2}(\underbrace{\deg_{\mathcal{S}_2}(\zeta_{\mathbb{F}}(x_1))}_{=0}) = B_{\mathcal{S}_1}(\deg_{\mathcal{S}_1}(\zeta_{\mathbb{F}}(x_1))) \ \cap \ B_{\mathcal{S}_2}(0).$$

It is an open neighbourhood of $\zeta_{\mathbb{F}}(x_1)$ in $Z_d(X)$. We have still to prove that for each open neighbourhood $W \Subset G$ of x_1, $\zeta_{\mathbb{F}}(W) \not\subset A$.
Let $W \Subset G$ be an open neighbourhood of x_1. We remark first that $R^{\mathbb{F}}(W) \cap R^{\mathbb{F}}(U_2) \neq \emptyset$, otherwise there is a contradiction with the choice of L_2, because $L_1 \subset R^{\mathbb{F}}(W)$ and $L_2 \subset R^{\mathbb{F}}(U_2)$. If $\widetilde{w} \in R^{\mathbb{F}}(W) \cap R^{\mathbb{F}}(U_2)$ then there exists $\widetilde{y} \in U_2$ such that $\widetilde{w} \in R^{\mathbb{F}}(\widetilde{y})$. Hence $\widetilde{y} \in R^{\mathbb{F}}(W) \cap U_2$ and thus $R^{\mathbb{F}}(W) \cap U_2 \neq \emptyset$. So we can choose $w \in W$ such that $R^{\mathbb{F}}(w) \cap U_2 \neq \emptyset$. Thus, $\zeta_{\mathbb{F}}(w) \notin B_{\mathcal{S}_2}(0)$ and so $\zeta_{\mathbb{F}}(W) \not\subset A$, which concludes the proof. ☐

5.4.6 Claim *The quotient $C/\mathbb{F}$ is Hausdorff.*

Proof We have the following equation

$$R^{\mathbb{F}}|_{C\times C} = \{(x,y) \in C \times C \mid \zeta_{\mathbb{F}}(x) = \zeta_{\mathbb{F}}(y)\} =: R.$$

Since $\zeta_{\mathbb{F}}$ is continuous on C and $Z_d(X)$ is Hausdorff (cf proposition 2.1.7), we see that $R = \overline{R}$, i.e. X/R is Hausdorff. $\square$

Question Does the equation $\{x \in G \mid \zeta_{\mathbb{F}} \text{ is continuous in } x\} = X^2_{\mathrm{st}}$ hold ?

The leaf space $X/\mathbb{F}$ and the meromorphic leaf space $Z(\mathbb{F})$

6 Leaf space and cycles

In this section we give new conditions that are equivalent to the fact that the leaf space $X/\mathbb{F}$ of a foliation $\mathbb{F}$ is a complex space. In a first subsection we prove the equivalence of these new conditions for regular foliations (theorem 6.1.1) and in the second subsection for non-regular foliations having leaves everywhere (theorem 6.2.1). Afterwards we explain how $X/\mathbb{F}$ can be interpreted as a subspace of $Z_d(X)$. In the last section we present an example of manifolds due to Hirzebruch. On these manifolds we construct foliations that illustrate theorem 6.1.1.

6.1 The main theorem on the leaf space of regular foliations

In this subsection we prove:

6.1.1 Theorem *If $\mathbb{F}$ is a d-dimensional regular holomorphic foliation with all leaves closed on a complex manifold X, then the following conditions are equivalent:*

(a) *$X/\mathbb{F}$ is a complex space*

(b) *$X/\mathbb{F}$ is Hausdorff*

(c) $X = X^1_{\mathrm{st}} = X^2_{\mathrm{st}}$

(d) *$G = X$ and $\zeta_{\mathbb{F}}$ is continuous*

(e) *There exists a continuous $\mathbb{F}$-invariant mapping $\varphi : X \longrightarrow Z_d(X)$ such that $|\varphi(x)| = L_x$ for each $x \in X$*

(f) *The canonical mapping $\psi : X \longrightarrow \widetilde{Z}_d(X)$ given by $\psi(x) = L_x$ is continuous.*

For a better understanding of the relation between φ and $\zeta_{\mathbb{F}}$ in theorem 6.1.1, we prove the following theorem:

6.1.2 Theorem *Let $\mathbb{F}$ be a d-dimensional regular holomorphic foliation with all leaves closed on a connected complex manifold X. If $\varphi\colon X \longrightarrow Z_d(X)$ is continuous and $\mathbb{F}$-invariant and if $|\varphi(x)| = L_x$ for each $x \in X$, then there exists a number $M \in \mathbb{N}_{>0}$ such that $\varphi(x) = M \cdot \zeta_{\mathbb{F}}(x)$ for each $x \in X$*

6.1.3 Example By lemma 6.4.16, the mapping $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ given by $f(z_1, z_2) = z_1^2 z_2^3$ is simple, i.e. all fibres of f are connected. Let $X := \mathbb{C}^2 \setminus \{0\}$ and let $\mathbb{F}$ be the regular foliation given by the restriction of f on X (see example 5.2.10). Here $G = X$. By example 5.2.10, the mapping $\zeta_{\mathbb{F}}\colon G \longrightarrow Z_1(X)$ is given by

$$\zeta_{\mathbb{F}}(z_1, z_2) = \begin{cases} [f^{-1}(f(z_1, z_2))] & \text{if } z_1 z_2 \neq 0 \\ 2L_1 & \text{if } z_1 = 0 \\ 3L_2 & \text{if } z_2 = 0 \end{cases}$$

where $L_1 = \{0\} \times \mathbb{C}^*$ and $L_2 = \mathbb{C}^* \times \{0\}$. This mapping is not continuous, because

$$\lim_{k \to \infty} \zeta_{\mathbb{F}} \left(\frac{1}{k}, 1 \right) = 2L_1 + 3L_2 \neq \zeta_{\mathbb{F}}(0, 1) = 2L_1.$$

Note that $X/\mathbb{F}$ is not Hausdorff.

Proof of theorem 6.1.1 (a) $\Leftrightarrow$ (b) is due to Holmann ([Hol72, Theorem 3.4]). (b) $\Leftrightarrow$ (c) is theorem 4.1.6.
"(a) $\Rightarrow$ (d)". By theorem 5.3.4, $X = G$. By remark 2.3.4,$(X/\mathbb{F})_{\mathrm{gen}(\pi)} = X/\mathbb{F}$. Hence the mapping $Z_\pi\colon X/\mathbb{F} \longrightarrow Z_d(X)$ is well-defined and continuous. By theorem 5.3.4, one sees that for each $x \in X$

$$(Z_\pi \circ \pi)(x) = \mu_a(L_x) \cdot L_x = \mu_t(L_x) \cdot L_x = \zeta_{\mathbb{F}}(x).$$

"(d) $\Rightarrow$ (e)". $\zeta_{\mathbb{F}}$ has the required properties.
"(e) $\Rightarrow$ (f)". Since the canonical projection $Z_d(X) \to \widetilde{Z}_d(X)$ is continuous and $\psi(x) = |\varphi(x)|$, ψ is continuous.
"(f) $\Rightarrow$ (b)". ψ induces an injective continuous mapping $\overline{\psi}\colon X/\mathbb{F} \longrightarrow \widetilde{Z}_d(X)$. Since $\widetilde{Z}_d(X)$ is Hausdorff, then $X/\mathbb{F}$ is also Hausdorff. $\square$

Proof of theorem 6.1.2 For each $x \in X$, there exists $n_x \in \mathbb{N}_{>0}$ such that $\varphi(x) = n_x L_x$. Let $x_0 \in X_{\mathrm{tr}}$ and (U, p) be a local $\mathbb{F}$-foliation at x_0. We can choose (U, p) such that $U \subset X_{\mathrm{tr}}$, $U \cap L_y$ is connected for each $y \in U$ and the scale $\mathcal{S}$ associated to (U, p) is adapted to $\varphi(y)$ for each $y \in U$ (If (U, p) is a local $\mathbb{F}$-foliation such that $U \cap L_y$ is connected for each $y \in U$, then we can shrink (U, p) such that the required condition is verified). Hence

$$\deg_{\mathcal{S}}(\varphi(y)) = n_y \quad \text{for each } y \in U.$$

Let $y \in \varphi^{-1}(B_{\mathcal{S}}(n_{x_0})) \cap U \Subset X_{\mathrm{tr}}$. Then

$$n_y = \deg_{\mathcal{S}}(\varphi(y)) = n_{x_0}.$$

Hence the mapping $x \mapsto n_x$ is locally constant on X_{tr}. By proposition 5.3.5, the set $X \setminus X_{\text{tr}}$ is analytically thin in X and consequently X_{tr} is connected. Hence $x \mapsto n_x$ is constant on X_{tr}. We denote this constant by M. If $y \in X \setminus X_{\text{tr}}$, we choose a sequence $y_\nu \to y$, $y_\nu \in X_{\text{tr}}$. By continuity of φ, $\varphi(y_\nu) \to \varphi(y)$. Moreover, $M \cdot \zeta_{\mathbb{F}}(y_\nu) \to M \cdot \zeta_{\mathbb{F}}(y)$. This proves that $\varphi(y) = M \cdot \zeta_{\mathbb{F}}(y)$. □

6.1.4 Proposition *If $\mathbb{F}$ is a compact regular foliation, then the two mappings $\zeta_{\mathbb{F}} : G \longrightarrow C_d(X)$ and $\psi|_G : G \longrightarrow \widetilde{C}_d(X)$ are continuous.*

Proof By theorem 5.2.12, if L is a leaf in G, then the holonomy group $H(L)$ of L has the order $\mu_t(L) < \infty$. Hence, by [Hol78, Proposition 4.2], $G/\mathbb{F}$ is Hausdorff. By theorem 6.1.1, $\zeta_{\mathbb{F}}$ and $\psi|_G$ are continuous. □

6.2 The main theorem on the leaf space of non-regular foliations

In this subsection, we consider coherent holomorphic foliations. If these foliations have leaves everywhere and are generically regular, we can prove a similar theorem as theorem 6.1.1:

6.2.1 Theorem *Let $\mathbb{F}$ be a d-dimensional open and generically regular foliation with leaves everywhere on a complex manifold X. If, in addition, $L \cap \operatorname{Sing} \mathbb{F} \neq L$ for each leaf L of $\mathbb{F}$, then the following conditions are equivalent:*

(a) *$X/\mathbb{F}$ is a complex space*

(b) *$\mathbb{F}$ is proper and there exists a continuous $\mathbb{F}$-invariant mapping $\varphi : X \longrightarrow Z_d(X)$ such that $|\varphi(x)| = L_x$ for each $x \in X$*

6.2.2 Remark The assumptions used in theorem 6.2.1 are the same assumptions that we can find for example in [HKR98, Proposition 2]. The question is if the result is true without some of this assumptions.

The main tool of the proof is a theorem of Grauert on semi-proper equivalence relations.

6.2.3 Definition An equivalence relation R on X is called ***semi-proper*** if the canonical projection $\pi : X \longrightarrow X/R$ is semi-proper, i.e. $\forall x \in X$, there exists a compact subset $K \subset X$ containing x such that $\pi(K)$ is a neighbourhood of $\pi(x)$.

6.2.4 Theorem ([Kau93, 1.6] due to Grauert) *If R is an equivalence relation on a maximal complex space X, then the following conditions are equivalent:*

(a) *X/R is a complex space*

(b) *R is semi-proper, X/R is Hausdorff and locally holomorphically separable*[8]. $\square$

To be able to use this theorem we prove

6.2.5 Lemma *Open equivalence relations are semi-proper.*

Proof Let R be an open equivalence relation. If $x \in X$, then we can choose a relatively compact open neighbourhood U of x in X. By the openness of R, $\pi(U)$ is open in X/R, where $\pi : X \longrightarrow X/R$ is the canonical projection. Hence $\pi(\overline{U})$ is a neighbourhood of $\pi(x)$ in X/R. $\square$

For the following construction, we suppose that $\mathbb{F}$ is open, but that in general it does not have leaves everywhere.

6.2.6 Definition The ***good-set*** $G(\mathbb{F})$ of $\mathbb{F}$ and the ***trivial locus*** X_{tr} of $\mathbb{F}$ are defined by

$$G(\mathbb{F}) := G(\mathbb{F}^{\text{ns}}) \quad \text{and} \quad X_{\text{tr}}(\mathbb{F}) := X_{\text{tr}}(\mathbb{F}^{\text{ns}}).$$

We write G, resp. X_{tr}, if it is not necessary to precise the foliation.

We consider the mapping $\zeta_{\mathbb{F}} : G \longrightarrow Z_d(X^{\text{ns}})$ given by $\zeta_{\mathbb{F}}(x) := \mu_t(L_x)[L_x]$, where $\mu_t(L_x)$ is the topological multiplicity of the $\mathbb{F}^{\text{ns}}$-leaf L_x. This mapping is well-defined since L_x is analytic in X^{ns} (see proposition 5.2.6 and lemma 3.1.7).

6.2.7 Remark If $x \in G$, then L_x is closed in X^{ns}. But L_x is not closed in X in general (see example 7.4.1). Hence, the cycle $\zeta_{\mathbb{F}}(x) \in Z_d(X^{\text{ns}})$ is not an element of $Z_d(X)$ in general, since L_x is not an analytic subset of X in general. If $\mathbb{F}$ is proper, then L_x is an analytic subset of X. Hence, in this case, $\zeta_{\mathbb{F}}(x)$ can be interpreted as an element of $Z_d(X)$.

6.2.8 Proposition *If $\mathbb{F}$ is an open foliation, then G and X_{tr} have the following properties:*

(a) *G is $\mathbb{F}$-saturated and open in X*

(b) *X_{tr} is $\mathbb{F}$-saturated and open in X*

In addition, if $\mathbb{F}$ is generically regular and if $\mathbb{F}^{\text{ns}}$ has all leaves closed (in X^{ns}), then

(c) *G and X_{tr} are dense in X.*

Proof This is a consequence of remark 3.2.12, proposition 5.2.6, proposition 5.2.16 and theorem 5.2.17. $\square$

[8]A ringed space $(X, \mathcal{O})$ is called **locally holomorphically separable** if for each $x \in X$, there exists an open neighbourhood U of x such that the functions in $\mathcal{O}(U)$ separate the points of U.

6.2.9 Lemma *Let $\mathbb{F}$ be a proper foliation on X. If there exists a continuous and $\mathbb{F}$-invariant mapping $\varphi : X \longrightarrow Z_d(X)$ such that $|\varphi(x)| = L_x$ for each $x \in X$, then $X/\mathbb{F}$ is Hausdorff.*

Proof Like (e)⇒(f)⇒(b) in the proof of theorem 6.1.1. □

Now we are prepared for the

Proof of theorem 6.2.1 (a)⇒(b) Since π is open and $X/\mathbb{F}$ is normal, $(X/\mathbb{F})_{\mathrm{gen}(\pi)} = X/\mathbb{F}$ by remark 2.3.4, and thus $Z_\pi : X/\mathbb{F} \longrightarrow Z_d(X)$ is well-defined and continuous. Hence $\varphi := Z_\pi \circ \pi$ is $\mathbb{F}$-invariant and continuous. Moreover, for each $x \in X$,

$$|\varphi(x)| = |Z_\pi(\pi(x))| \underset{\substack{\uparrow \\ \text{by remark 2.3.6}}}{=} \pi^{-1}(\pi(x)) = L_x.$$

Ad (b)⇒(a): The equivalence relation $R^{\mathbb{F}}$ is semi-proper by lemma 6.2.5. Furthermore $X/\mathbb{F}$ is Hausdorff by lemma 6.2.9. By theorem 6.2.4 it suffices to prove that $X/\mathbb{F}$ is locally holomorphically separable, i.e. each leaf L has an open $\mathbb{F}$-saturated neighbourhood $Y \Subset X$ such that for each distinct leaves L_1 and L_2 of $\mathbb{F}$ in Y, there exists an $\mathbb{F}$-invariant holomorphic function $f \in \mathcal{O}(Y)$ such that $f(L_1) \neq f(L_2)$.
Let L be an $\mathbb{F}$-leaf. By the assumption that $L \cap \operatorname{Sing} \mathbb{F} \neq L$ and since $\mathbb{F}$ is proper, there exists a local $\mathbb{F}$-foliation (U, p) of $\mathbb{F}^{\mathrm{reg}}$ such that $\emptyset \neq L \cap U$ is connected. We restrict U such that the scale $\mathcal{S}$ associated to (U, p) is adapted to the cycle $\varphi(L)$. Let Y be the connected component of $\varphi^{-1}\Big(B_{\mathcal{S}}(\deg_{\mathcal{S}}(\varphi(L)))\Big)$ that contains x. The set Y is an open $\mathbb{F}$-saturated neighbourhood of L.
Let L_1 and L_2 be two distinct leaves of $\mathbb{F}$ in Y. By the definition of Y, $\varphi(L_j) \in B_{\mathcal{S}}(\deg_{\mathcal{S}}(\varphi(L)))$. By lemma 2.1.17, there exists a holomorphic function $g \in \mathcal{O}(U)$ and $t \in W := q(U)$ such that $g^\sharp_t(\varphi(L_1)) \neq g^\sharp_t(\varphi(L_2))$. By definition of Y, $\varphi(L) \in B_{\mathcal{S}}(\deg_{\mathcal{S}}(\varphi(L)))$ for each $L \subset Y$. Hence

$$f := g^\sharp_t \circ \varphi : Y \longrightarrow \mathbb{C}$$

is well-defined. By construction, $f(L_1) \neq f(L_2)$. By lemma 2.1.16 f is continuous and since φ is $\mathbb{F}$-invariant, f is $\mathbb{F}$-invariant. We have to show that f is holomorphic on Y.
Since $|\varphi(x)|$ is analytic in X^{ns} for each $x \in X^{\mathrm{ns}}$ (because $\mathbb{F}$ is proper), the cycle $\varphi(x) \in Z_d(X)$ can be interpreted as a cycle in $Z_d(X^{\mathrm{ns}})$. We obtain a mapping $\widetilde{\varphi} : X^{\mathrm{ns}} \longrightarrow Z_d(X^{\mathrm{ns}})$. By our assumptions on φ, the mapping $\widetilde{\varphi}$ satisfies the condition (e) of theorem 6.1.1 (it is continuous since each scale in X^{ns} is a scale in X). Thus $G = X^{\mathrm{ns}}$ and $X^{\mathrm{ns}}/\mathbb{F}^{\mathrm{ns}}$ is a complex space. Moreover by theorem 6.1.2, there exists $M \in \mathbb{N}_{>0}$ such that $\widetilde{\varphi}(x) = M \cdot \zeta_{\mathbb{F}^{\mathrm{ns}}}(x) = M \cdot \zeta_{\mathbb{F}}(x)$ for each $x \in X^{\mathrm{ns}}$.
Let $x \in X_{\mathrm{tr}} \cap Y$. By the previous argumentation, $\widetilde{\varphi}(x) = M \cdot [L_x]$. Thus $\varphi(x) = M \cdot [L_x] \in Z_d(X)$. Hence

$$f(x) = M \cdot \sum_{y \in U_t \cap L_x} g(y),$$

where $U_t = q^{-1}(t)$. Furthermore, since $\varphi(x) \in B_{\mathcal{S}}(\deg_{\mathcal{S}}(\varphi(L)))$,

$$\begin{aligned} \deg_{\mathcal{S}}(\varphi(L)) &= \deg_{\mathcal{S}}(\varphi(x)) \\ &= M \cdot (\text{number of sheets of } L_x \cap U \to W) \\ &= M \cdot \mathrm{Card}(U_t \cap L_x). \end{aligned}$$

Hence $c := \mathrm{Card}(U_t \cap L_x)$ is independent of the choice of $x \in X_{\mathrm{tr}} \cap Y$
Let $x \in X_{\mathrm{tr}} \cap Y$. We note $U_t \cap L_x =: \{x =: x_1, x_2, \ldots, x_c\}$. For each $j = 1, \ldots, c$, there exists an open neighbourhood V_j of x_j in $U_t \cap X_{\mathrm{tr}} \cap Y$ such that

- $V_j \cap V_k = \emptyset$ for each $j \neq k$
- For each j and for each leaf L' in Y, $V_j \cap L'$ is empty or contains exactly one point (since $x_j \in X_{\mathrm{tr}}$)
- For each j, there exists a biholomorphic mapping $h_j : V_1 \longrightarrow V_j$ such that $h_j(x')$ and x' belong to the same leaf for each $x' \in V_1$ (cf lemma 3.1.6).

By the property of c, for each $x' \in V_1$, $L_{x'} \cap U_t = \{x', h_2(x'), \ldots, h_c(x')\}$. Thus for each $x' \in V_1$,

$$f(x') = M \cdot \sum_{y \in U_t \cap L_{x'}} g(y) = M \cdot \sum_{k=1}^{c} g(h_k(x')).$$

Hence $f|_{V_1} \equiv M \cdot \sum_{k=1}^{c} (g|_{V_k}) \circ h_k$, and thus $f|_{V_1}$ is holomorphic. Since (U, p) is a local foliation, $f|_{p^{-1}(p(V_1))}$ is holomorphic.
Our construction shows that $f|_{X_{\mathrm{tr}} \cap Y}$ is locally the sum of a finite number of holomorphic functions. Thus, $f \in \mathcal{O}(X_{\mathrm{tr}} \cap Y)$. By proposition 5.3.5 and since $X^{\mathrm{ns}}/\mathbb{F}^{\mathrm{ns}}$ is a complex space, $X^{\mathrm{ns}} \setminus X_{\mathrm{tr}}$ is analytically thin in X^{ns}. Since f is continuous on Y, $f \in \mathcal{O}(X^{\mathrm{ns}} \cap Y)$ by the first Riemann removable singularity theorem (see for example [KK83, Theorem 7.6] or [Fis76, 2.23]). Furthermore $X \setminus X^{\mathrm{ns}}$ is analytically thin in X, because $\mathbb{F}$ is generically regular. Thus, by the same theorem, $f \in \mathcal{O}(Y)$, which completes the proof. □

6.3 The leaf space as subspace of $Z_d(X)$

In this subsection, $\mathbb{F}$ denotes a singular holomorphic foliation.

6.3.1 Theorem *If $\mathbb{F}$ is a d-dimensional proper and open foliation on a complex manifold X of dimension n and if $\varphi : X \longrightarrow Z_d(X)$ is a continuous $\mathbb{F}$-invariant mapping such that $|\varphi(x)| = L_x$ for each $x \in X$, then the mapping $\overline{\varphi} : X/\mathbb{F} \longrightarrow \varphi(X)$ induced by φ is a homeomorphism (see the commutative diagram 6.3.2).*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"] & \varphi(X) \subset Z_d(X) \\ X/\mathbb{F} \arrow[ur, "\overline{\varphi}"] & \end{tikzcd} \tag{6.3.2}$$

6.3.3 Theorem *Let $\mathbb{F}$ be a d-dimensional compact open foliation, on a complex manifold X of dimension n, such that $X/\mathbb{F}$ is a complex space. Then*

- *The mapping $Z_\pi : X/\mathbb{F} \longrightarrow C_d(X)$ is well-defined and is a homeomorphism onto its image*
- *Z_π is a proper analytic family*
- *$Z_\pi : X/\mathbb{F} \longrightarrow \mathcal{B}_d(X)$ is holomorphic.*

6.3.4 Corollary *$C_d(X)$ and $\mathcal{B}_d(X)$ induce the same topology on $Z_\pi(X)$.*

Proof Let $A := Z_\pi(X) \subset \mathcal{B}_d(X)$ with the induced topology and $B := Z_\pi(X) \subset C_d(X)$ with the induced topology. We have to see that the identity mapping $A \to B$ is a homeomorphism. By remark 2.2.7, $A \to B$ is continuous. Furthermore, by theorem 6.3.3, $Z_\pi : X/\mathbb{F} \longrightarrow B$ is a homeomorphism and $Z_\pi : X/\mathbb{F} \longrightarrow A$ is continuous (even holomorphic). Hence the identity mapping $B \to A$ is continuous. $\square$

6.3.5 Claim *Under the assumptions of theorem 6.3.1, $\varphi : X \longrightarrow \varphi(X) \subset Z_d(X)$ is open.*

Proof Let $U \Subset X$ and let $x \in U$. We have to find an open neighbourhood $A \Subset \varphi(X)$ of $\varphi(x)$ such that $A \subset \varphi(U)$. Let $\mathcal{S} := (\chi : V \longrightarrow \Omega, W, D)$ be a scale adapted to $\varphi(x)$ such that $x \in |\mathcal{S}|$. We choose $\mathcal{S}$ such that $|\mathcal{S}| \subset U$. We set $A := B_{\mathcal{S}}(\deg_{\mathcal{S}}(\varphi(x))) \cap \varphi(X)$. It is an open neighbourhood of $\varphi(x)$ in $\varphi(X)$. If $Z \in A$, then there exists $y \in X$ such that $\varphi(y) = Z$. Since $\deg_{\mathcal{S}}(\varphi(y)) = \deg_{\mathcal{S}}(\varphi(x)) \neq 0$,

$$|\mathcal{S}| \cap L_y = |\mathcal{S}| \cap |\varphi(y)| \neq \emptyset.$$

Thus, there exists $y' \in |\mathcal{S}| \cap L_y \subset U$. Hence $Z = \varphi(y) = \varphi(y') \in \varphi(U)$. It follows that $A \subset \varphi(U)$ which concludes the proof. $\square$

Proof of theorem 6.3.1 Since φ is $\mathbb{F}$-invariant, $\overline{\varphi}$ is well-defined. Furthermore, $\overline{\varphi}$ is clearly surjective. If $x, x' \in X$ such that $L_x \neq L_{x'}$, then $\varphi(x) \neq \varphi(x')$, because $|\varphi(x)| = L_x$ and $|\varphi(x')| = L_{x'}$. Hence $\overline{\varphi}$ is injective. Since the topology on $X/\mathbb{F}$ is the quotient topology, $\overline{\varphi}$ is continuous. By claim 6.3.5, $\overline{\varphi}$ is open. $\square$

Proof of theorem 6.3.3 By lemma 2.5.4, π is geometrically flat, since π is open and surjective and $X/\mathbb{F}$ is normal,. Hence Z_π is well-defined. By theorem 6.3.1, Z_π is a homeomorphism onto its image.
By corollary 2.5.6, $Z_\pi : X/\mathbb{F} \longrightarrow Z_d(X)$ is an analytic family. Furthermore, since $X/\mathbb{F}$ is Hausdorff, each leaf of $\mathbb{F}$ has a fundamental system of saturated open neighbourhoods (Similar to the proof of proposition 5.1.3). Hence Z_π is a proper analytic family of cycles.
By the Barlet's theorem (theorem 2.2.6), $Z_\pi : X/\mathbb{F} \longrightarrow \mathcal{B}_d(X)$ is holomorphic. $\square$

6.4 Example of foliations according to a construction of Hirzebruch ([Hir53])

In this subsection, we present an example of singular foliations of codimension one with a singular leaf composed of several irreducible components, each with a well-defined multiplicity. This example illustrates theorem 6.2.1.

We give two numbers n and q such that $0 < q < n$ and n and q are relatively prime numbers. The aim is to construct a 2-dimensional manifold $H(n,q)$ and a simple open mapping $f_{n,q} : H(n,q) \longrightarrow \mathbb{C}$ which is a global integral of a coherent holomorphic foliation $\mathbb{F}_{n,q}$ of codimension 1 on $H(n,q)$.

The idea of the construction is based on the second part of the thesis of Hirzebruch (see [Hir53]).

We apply an algorithm similar to the Euclide's one on the numbers n and q. We set $\lambda_0 := n$, $\lambda_1 := q$. By the following sequence of operations we find a number s and the lists $(\lambda_0, \dots, \lambda_{s+1}) \in \mathbb{N}^{s+2}$ and $(b_1, \dots, b_s) \in \mathbb{N}^s$:

$$
\begin{array}{llll}
\text{find } b_1 \text{ and } \lambda_2 \text{ such that} & \lambda_0 = b_1\,\lambda_1 - \lambda_2 & \text{and} & 0 \leq \lambda_2 < \lambda_1 \\
\vdots & \vdots & & \vdots \\
\text{find } b_k \text{ and } \lambda_{k+1} \text{ such that} & \lambda_{k-1} = b_k\,\lambda_k - \lambda_{k+1} & \text{and} & 0 \leq \lambda_{k+1} < \lambda_k \\
\vdots & \vdots & & \vdots \\
\text{find } b_s \text{ and } \lambda_{s+1} \text{ such that} & \lambda_{s-1} = b_s\,\lambda_s - \lambda_{s+1} & \text{and} & 0 = \lambda_{s+1} < \lambda_s = 1.
\end{array}
$$

By construction, $b_k > 1$ for $1 \leq k \leq s$.

6.4.1 Remark Since n and q are relatively prime numbers, if $q \neq 1$ then $\lambda_2 \neq 0$ and thus s is greater than 1.

6.4.2 Remark The list $(b_1, \dots, b_s)$ can be calculated representing n/q in the following special continuous fraction:

$$
\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots - \cfrac{1}{b_{s-1} - \cfrac{1}{b_s}}}}}
$$

6.4.3 Remark The list $(\lambda_0, \dots, \lambda_{s+1})$ can be calculated recursively from the list $(b_1, \dots, b_s)$:

$$
\lambda_0 := n \qquad \lambda_1 := q \qquad \lambda_{k+1} := b_k \lambda_k - \lambda_{k-1}.
$$

We construct the lists $(\mu_0, \dots, \mu_{s+1})$ and $(\nu_0, \dots, \nu_{s+1})$ in the following way:

$$
\mu_0 := 0 \qquad \mu_1 := 1 \qquad \mu_{k+1} := b_k\,\mu_k - \mu_{k-1}
$$

$$\nu_0 := 1 \qquad \nu_1 := 1 \qquad \nu_{k+1} := b_k\,\nu_k - \nu_{k-1}.$$

6.4.4 Proposition

(a) $\lambda_k + (n-q)\mu_k = n\,\nu_k$, *for* $0 \leq k \leq s+1$.

(b) $\lambda_k\,\mu_{k+1} - \lambda_{k+1}\,\mu_k = n$, *for* $0 \leq k \leq s$.

(c) $\nu_k\,\mu_{k+1} - \nu_{k+1}\,\mu_k = 1$, *for* $0 \leq k \leq s$.

(d) $0 < \mu_k < \mu_{k+1}$, *for* $0 \leq k \leq s$, *and* $\mu_{s+1} = n$.

(e) $0 < \nu_k \leq \nu_{k+1}$, *for* $0 \leq k \leq s$, *and* $\nu_{s+1} = n-q$.

Ad(a) The equation is correct for $k=0$ and $k=1$. We suppose it is correct for $k-1$ and k, and calculate

$$\begin{aligned} n\,\nu_{k+1} &= n\,b_k\nu_k - n\,\nu_{k-1} \\ &= b_k(\lambda_k + (n-q)\mu_k) - (\lambda_{k-1} + (n-q)\mu_{k-1}) \\ &= \lambda_{k+1} + (n-q)\mu_{k+1}. \end{aligned}$$

Ad(b) The equation is correct for $k=0$. We suppose it is correct for $k-1$ and calculate

$$\begin{aligned} \lambda_k\,\mu_{k+1} - \lambda_{k+1}\,\mu_k &= \lambda_k(b_k\,\mu_k - \mu_{k-1}) - \mu_k(b_k\,\lambda_k - \lambda_{k-1}) \\ &= \lambda_{k-1}\,\mu_k - \lambda_k\,\mu_{k-1} \;=\; n. \end{aligned}$$

Ad(c) The proof is similar as (b)'s one.

Ad(d) For $k=0$, $\mu_0 = 0 < 1 = \mu_1$. We suppose that the inequality is correct for $k-1$ and calculate

$$\begin{aligned} \mu_{k+1} - \mu_k &= b_k\,\mu_k - \mu_{k-1} - \mu_k \\ &= \underbrace{(b_k - 2)}_{\geq 0}\mu_k + \underbrace{\mu_k - \mu_{k-1}}_{>0} > 0 \end{aligned}$$

By (b) and since $\lambda_{s+1} = 0$, the equation $\mu_{s+1} = n$ holds.

Ad(e) The proof is similar as (d)'s one. □

6.4.5 Lemma *For* $0 \leq k \leq s$, *the numbers* ν_k *and* ν_{k+1}, *resp. the numbers* μ_k *and* μ_{k+1}, *are relatively prime numbers.*

Proof We apply to equation (c) of proposition 6.4.4 a well-known result of number theory, called "Bezout Identity", saying that two numbers a and b are relatively prime numbers iff there exist two numbers s and t such that $as + bt = 1$ (see [Bou64, VII.1.2, Théorème 1] or [RU95, 2.1.5]). □

We construct now $H(n,q)$. We take $s+1$ copies of $\mathbb{C}^2$ and denote them by $H_0, \ldots, H_s$. The coordinates of H_k are denoted by (u_k, v_k). We define

$$H'_k := H_k \setminus \{u_k = 0\} \qquad H''_k := H_k \setminus \{v_k = 0\}.$$

For $0 \le k < s$, the mapping $\varphi_k : H'_k \longrightarrow H''_{k+1}$ given by

$$\varphi_k(u_k, v_k) = \left(u_k^{b_{k+1}}\, v_k, \frac{1}{u_k} \right)$$

is biholomorphic, and its inverse is given by

$$\varphi_k^{-1}(u_{k+1}, v_{k+1}) = \left(\frac{1}{v_{k+1}}, u_{k+1}(v_{k+1})^{b_{k+1}} \right).$$

We define $\widetilde{H} := \coprod_{k=0}^{s} H_k$ and $H(n,q) := \widetilde{H}/\sim$ where $\sim$ is the equivalence relation generated by

$$(u_k, v_k) \sim \varphi_k(u_k, v_k) \quad \forall\ (u_k, v_k) \in H'_k, \quad 0 \le k < s.$$

We denote the canonical projection by $h : \coprod H_k \longrightarrow H(n,q)$. For $0 \le k \le s$, we define the mappings $h_k : H_k \longrightarrow H(n,q)$ to be the restriction of h on H_k. For each $(u_k, v_k) \in H'_k$, the equation

$$h_k(u_k, v_k) = h_{k+1}(\underbrace{\varphi_k(u_k, v_k)}_{\in H''_{k+1}}) \tag{6.4.6}$$

holds.

6.4.7 Lemma *$h_j(H'_j \cap H''_j) = h_k(H'_k \cap H''_k)$, for $0 \le j, k \le s$.*

Proof We suppose that $j < k$.
For "$\subset$", let $(u_j, v_j) \in H'_j \cap H''_j$. If we write

$$(u_k, v_k) := (\varphi_{k-1} \circ \cdots \circ \varphi_j)(u_j, v_j),$$

then $(u_k, v_k) \in H'_k \cap H''_k$ and $h_k(u_k, v_k) = h_j(u_j, v_j)$, by equation 6.4.6. Thus $h_j(u_j, v_j) \in h_k(H'_k \cap H''_k)$. The proof of "$\supset$" is similar. □

6.4.8 Theorem *$H(n,q)$ is a complex manifold of dimension 2. For $0 \le k \le s$, the mapping $h_k : H_k \longrightarrow h_k(H_k) \Subset H(n,q)$ is a local parametrisation of $H(n,q)$ and $\{h_k(H_k), 0 \le k \le s\}$ is an atlas of $H(n,q)$. The transition mappings are given by compositions of some $\varphi_0, \ldots, \varphi_{s-1}$.*

For the proof, we consider the following

6.4.9 Lemma *Let N_1, N_2 be two complex manifolds, let $U_j \Subset N_j$ be an open subset of N_j ($j = 1, 2$), and let $\varphi : U_1 \longrightarrow U_2$ be biholomorphic. We denote $N_1 \amalg_\varphi N_2 := (N_1 \amalg N_2)/\sim$, where $\sim$ is the equivalence relation generated by $n_1 \sim \varphi(n_1)$ for each $n_1 \in U_1$. Let $\pi : N_1 \amalg N_2 \longrightarrow N_1 \amalg_\varphi N_2$ be the canonical projection. Then π is an open mapping. If, in addition, the condition*

$$\text{for each } x_1 \in \partial U_1 \text{ and each } x_2 \in \partial U_2 \text{ there exists an open neighbourhood } V_1 \Subset N_1, \text{ resp. } V_2 \Subset N_2, \text{ of } x_1, \text{ resp. of } x_2, \text{ such that } \varphi(V_1 \cap U_1) \cap V_2 = \emptyset \tag{6.4.10}$$

is satisfied, then $N_1 \amalg_\varphi N_2$ is Hausdorff, and the mappings $\psi_j := \pi|_{N_j} : N_j \longrightarrow \pi(N_j)$, $j = 1, 2$, are isomorphisms of ringed spaces (i.e. $N_1 \amalg_\varphi N_2$ is a complex manifold).

Proof We prove first the openness of π. We prove that the equivalence relation is open, i.e. if $U \Subset N_1 \amalg N_2$, then the saturated hull of U is open. We can suppose that $U \Subset N_1$. If $U \cap U_1 = \emptyset$, then the saturated hull of U is U, that is open. If $U \cap U_1 \neq \emptyset$, then the saturated hull of U is $U \cup \varphi(U \cap U_1)$, that is open. These two cases prove the openness of π.
To show that $N_1 \amalg_\varphi N_2$ is Hausdorff, we have to prove that for each $x_1, x_2 \in N_1 \amalg N_2$ such that $x_1 \not\sim x_2$, there exists a saturated open subset V_1', resp. V_2', of x_1, resp. x_2, such that $V_1' \cap V_2' = \emptyset$. The only case that is not trivial is if $x_j \in \partial U_j$, $j = 1, 2$. In this case, there exists an open neighbourhood $V_1 \Subset N_1$, resp. $V_2 \Subset N_2$, of x_1, resp. x_2, such that $\varphi(V_1) \cap V_2 = \emptyset$ and $V_1 \cap \varphi^{-1}(V_2) = \emptyset$. Thus, the sets $V_1' := (V_1 \cup \varphi(V_1 \cap U_1))$ and $V_2' := (V_2 \cup \varphi^{-1}(V_2 \cap U_2))$ are open and saturated, and

$$\begin{aligned} V_1' \cap V_2' &= (V_1 \cap V_2) \cup (V_1 \cap \varphi^{-1}(V_2 \cap U_2)) \cup (V_2 \cap \varphi(V_1 \cap U_1)) \cup \\ &\quad \cup (\varphi(V_1 \cap U_1) \cap \varphi^{-1}(V_2 \cap U_2)) \\ &= \emptyset. \end{aligned}$$

The mapping ψ_1 is clearly bijective, continuous and a homomorphism of ringed spaces. Since, for each $U \Subset N_1$,

$$\pi^{-1}(\pi(U) \cap \pi(N_1)) = U \amalg \varphi(U \cap U_1)$$

is open in $N_1 \amalg N_2$, ψ_1 is an open mapping. Let $U \Subset N_1$ and $f \in {}_{N_1}\mathcal{O}(U)$. The mapping $\widetilde{f} : U \amalg \varphi(U \cap U_1) \longrightarrow \mathbb{C}$ given by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in U \\ (f \circ \varphi^{-1})(x) & \text{if } x \in \varphi(U \cap U_1). \end{cases}$$

is an invariant holomorphic mapping. Thus ψ_1 is an isomorphism of ringed spaces.
□

Proof of theorem 6.4.8 By induction, for $0 \leq k \leq s-1$, we construct the space $\widetilde{H}_k$, the injective mapping $\psi_k : H_k \longrightarrow \widetilde{H}_k$, the open subset $\widetilde{H}_k' := \psi_k(H_k') \Subset \widetilde{H}_k$ and the mapping $\alpha_k : \widetilde{H}_k' \longrightarrow H_{k+1}''$.
For $k = 0$, we set $\widetilde{H}_0 := H_0$, $\widetilde{H}_0' := H_0'$, $\psi_0 := \mathrm{Id}$ and $\alpha_0 := \varphi_0$.

For $0 < k < s$, we define $\widetilde{H}_{k+1} := \widetilde{H}_k \amalg_{\alpha_k} H_{k+1}$ and ψ_{k+1} to be the canonical injection of H_{k+1} in $\widetilde{H}_{k+1}$. The following diagram shows the situation:

$$\begin{tikzcd}
H_k \arrow[d, "\psi_k"] & \supset & H'_k \arrow[r, "\varphi_k"] \arrow[d] & H''_{k+1} & \subset & H_{k+1} \arrow[d, "\psi_{k+1}"] \\
\widetilde{H}_k & \supset & \widetilde{H}'_k \arrow[ur, "\alpha_k"] & & & \widetilde{H}_{k+1} & = & \widetilde{H}_k \amalg_{\alpha_k} H_{k+1}
\end{tikzcd}$$

We define $\alpha_{k+1} := \varphi_{k+1} \circ \psi_{k+1}^{-1}$.
Finally, we define $\widetilde{H}_s := \widetilde{H}_{s-1} \amalg_{\alpha_{s-1}} H_s$. To each step of the induction, condition (6.4.10) of lemma 6.4.9 is verified. Furthermore, by lemma 6.4.9, the spaces $\widetilde{H}_k$ are manifolds, the mappings $\psi_k|_{H'_k} : H_k \longrightarrow \widetilde{H}'_k$ are biholomorphic and α_k are well-defined. The equation $H(n,q) = \widetilde{H}_s$ holds. The conclusions of the theorem are consequences of the construction. $\square$

We give s subsets $\sigma_1, \dots, \sigma_s$ of $H(n,q)$ by

$$\sigma_k := h_{k-1}\Big(\{(u_{k-1}, v_{k-1}) \in H_{k-1}, v_{k-1} = 0\}\Big) \cup h_k\Big(\{(u_k, v_k) \in H_k, u_k = 0\}\Big).$$

We denote

$$\sigma := \sigma_1 \cup \dots \cup \sigma_s.$$

6.4.11 Proposition *σ_k is a compact connected complex curve and σ is a connected analytic subset of $H(n,q)$.*

Proof The sets $h_{k-1}(\{v_{k-1} = 0\})$ and $h_k(\{u_k = 0\})$ are submanifolds of dimension 1 of $H(n,q)$. By equation 6.4.6, the equations

$$\begin{aligned}
h_{k-1}(\{v_{k-1} = 0\}) \cup h_k(\{u_k = 0\}) &= h_{k-1}(\{v_{k-1} = 0\}) \cup h_k(0,0) \\
&= h_k(\{u_k = 0\}) \cup h_{k-1}(0,0)
\end{aligned}$$

hold. Hence σ_k is a submanifold of dimension 1. By the previous equations, σ_k is connected.
For the compactness of σ_k, let $(U_j)_{j \in J}$ be an open cover of σ_k in $H(n,q)$. We can suppose that there exists $j_0 \in J$ such that $h_{k-1}(0,0) \in U_{j_0}$ and $h_k(0,0) \notin U_{j_0}$. Thus the set $V := (h_{k-1})^{-1}(U_{j_0} \cap \sigma_k)$ is an open neighbourhood of $(0,0)$ in $\{v_{k-1} = 0\}$. Hence there exists a number $r > 0$ such that $B_r(0) \times \{0\} \subset V$ ($B_r(0) \subset \mathbb{C}$ is the disk of radius r and center 0). Thus

$$\begin{aligned}
h_k^{-1}(U_{j_0} \cap \sigma_k) &\supset \varphi_{k-1}(V \cap H'_{k-1}) \\
&\supset \varphi_{k-1}((B_r(0) \times \{0\}) \smallsetminus \{(0,0)\}) = \{0\} \times \{|u_k| > r\}.
\end{aligned}$$

Thus there exists a compact $K \subset \mathbb{C}$ such that

$$\{u_k = 0\} \smallsetminus h_k^{-1}(U_{j_0} \cap \sigma_k) \subset K,$$

and thus $\sigma_k \setminus (U_{j_0} \cap \sigma_k) \subset h_k(K)$ which is compact. Thus σ_k is compact.
σ is a finite union of complex manifolds. Thus σ is analytic in $H(n,q)$. By definition of σ_k,

$$\sigma_j \cap \sigma_k = \begin{cases} \{h_j(0,0)\} & \text{if } j+1=k \\ \{h_k(0,0)\} & \text{if } j=k+1 \\ \emptyset & \text{if } |j-k| \neq 1. \end{cases} \tag{6.4.12}$$

Therefore σ is connected. □

We denote

$$\sigma_0 := h_0\Big(\{(u_0,v_0) \in H_0, u_0 = 0\}\Big) \quad \text{and} \quad \sigma_{s+1} := h_s\Big(\{(u_s,v_s) \in H_s, v_s = 0\}\Big).$$

6.4.13 Lemma *For $0 \leq k \leq s$, $H(n,q) \setminus (\sigma \cup \sigma_0 \cup \sigma_{s+1}) = h_k(H'_k \cap H''_k)$. In addition, if $s \geq 2$, then $\bigcap_{j=0}^{s} h_j(H_j) = H(n,q) \setminus (\sigma \cup \sigma_0 \cup \sigma_{s+1})$.*

Proof The equation $H(n,q) \setminus (\sigma \cup \sigma_0 \cup \sigma_{s+1}) = h_k(H'_k \cap H''_k)$ holds for $0 \leq k \leq s$ by definition of σ, σ_0 and σ_{s+1}, and by lemma 6.4.7.
Suppose that $s \geq 2$. For $0 \leq j < s$, $h_j(H_j) = h_j(H'_j \cap H''_j) \cup h_j(\{u_j = 0\}) \cup h_j(\{v_j = 0\})$. Thus, for $0 \leq j \leq s$,

$$\bigcap_{k=0}^{1} h_{j+k}(H_{j+k}) = \bigcap_{k=0}^{1} h_{j+k}(H'_{j+k} \cap H''_{j+k}) \cup h_j(\{v_j = 0\} \setminus \{(0,0)\}),$$

and for $0 \leq j < s-1$ (there exists a so-defined j because $s \geq 2$),

$$\bigcap_{k=0}^{2} h_{j+k}(H_{j+k}) = \bigcap_{k=0}^{2} h_{j+k}(H'_{j+k} \cap H''_{j+k}).$$

Thus, by lemma 6.4.7, $\bigcap_j h_j(H_j) = h_k(H'_k \cap H''_k)$, for $0 \leq k \leq s$. □

For $0 \leq k \leq s$, we define the mapping $f^{(k)}_{n,q} : H_k \longrightarrow \mathbb{C}$ by

$$f^{(k)}_{n,q}(u_k,v_k) := u_k^{\nu_k} v_k^{\nu_{k+1}}.$$

6.4.14 Claim *For $0 \leq k < s$ and $(u_k,v_k) \in H'_k$, the equation $f^{(k)}_{n,q}(u_k,v_k) = f^{(k+1)}_{n,q}(\varphi_k(u_k,v_k))$ holds.*

Proof We calculate

$$\begin{aligned} f^{(k+1)}_{n,q}(\varphi_k(u_k,v_k)) &= f^{(k+1)}_{n,q}\left(u_k^{b_{k+1}} v_k, \frac{1}{u_k}\right) \\ &= u_k^{b_{k+1}\,\nu_{k+1} - \nu_{k+2}}\, v_k^{\nu_{k+1}} \\ &= u_k^{\nu_k} v_k^{\nu_{k+1}} \\ &= f^{(k)}_{n,q}(u_k,v_k). \end{aligned}$$

□

We define
$$f_{n,q}\colon H(n,q) \longrightarrow \mathbb{C}$$
by $f_{n,q}(x) := f_{n,q}^{(k)}(u_k, v_k)$ if $h_k(u_k, v_k) = x$. This mapping is well-defined by claim 6.4.14 and equation 6.4.6

6.4.15 Proposition *$f_{n,q}$ is surjective, open and simple*

Proof For $0 \le k \le s$, $f_{n,q}^{(k)}(H_k) = \mathbb{C}$. Thus $f_{n,q}^{(k)}$ is surjective, and thus $f_{n,q}$ is surjective.
Since $f_{n,q}$ is not constant, $f_{n,q}$ is open, because each non-constant function on a complex manifold is open.
The fibre $f_{n,q}^{-1}(0) = \sigma \cup \sigma_0 \cup \sigma_{s+1}$ is connected. By lemma 6.4.16, the mappings $f_{n,q}^{(k)}$ are simple, because ν_k and ν_{k+1} are relatively prime numbers (see lemma 6.4.5). By lemma 6.4.13, for each $c \neq 0$
$$f_{n,q}^{-1}(c) \subset H(n,q) \setminus f_{n,q}^{-1}(0) = h_k(H'_k \cap H''_k). \quad (0 \le k \le s)$$
Thus, for each k, $f_{n,q}^{-1}(c) = h_k((f_{n,q}^{(k)})^{-1}(c))$, which is connected. □

6.4.16 Lemma *The mapping $f\colon\mathbb{C}^2 \longrightarrow \mathbb{C}$ given by $f(z_1, z_2) = z_1^{k_1} z_2^{k_2}$, where k_1 and k_2 are relatively prime numbers, is simple.*

Proof The fibre $f^{-1}(0) = (\{0\} \times \mathbb{C}) \cup (\mathbb{C} \times \{0\})$ is connected. Let $c \neq 0$. We have to show that $f^{-1}(c) = \{z \in \mathbb{C}^2 \mid z_1^{k_1} z_2^{k_2} = c\}$ is connected. We define $\gamma\colon\mathbb{C}^* \longrightarrow \mathbb{C}^2$ by $\gamma(t) := (t^{k_2}, c'/t^{k_1})$, where c' is a fixed complex number such that $(c')^{k_2} = c$. Then
$$f(\gamma(t)) = t^{k_1 k_2} \left(\frac{c'}{t^{k_1}}\right)^{k_2} = t^{k_1 k_2} \frac{c}{t^{k_1 k_2}} = c.$$
Thus $\gamma(\mathbb{C}^*) \subset f^{-1}(c)$. If $\gamma(\mathbb{C}^*) = f^{-1}(c)$, then $f^{-1}(c)$ is connected, because $\mathbb{C}^*$ is connected and γ is continuous. Hence we have to prove that $\gamma(\mathbb{C}^*) = f^{-1}(c)$.
First, note that if z_1 is fixed, then the equation $z_1^{k_1} x^{k_2} = c$ has exactly k_2 solutions. Let z_1 be fixed, and we calculate how many x are solution of $\gamma(t) = (z_1, x)$. We calculate
$$\gamma(t) = (z_1, x) \quad \Longleftrightarrow \quad \left\{ \begin{array}{lcl} t^{k_2} & = & z_1 \\ t^{k_1} & = & \frac{c'}{x}. \end{array} \right.$$
Denote $z_1 = re^{i\varphi}$. Then
$$t = \sqrt[k_2]{r} \exp\left(i\ \tfrac{\varphi + 2l\pi}{k_2}\right) \quad \text{and} \quad t^{k_1} = (\sqrt[k_2]{r})^{k_1} \exp\left(i\ \tfrac{k_1 \varphi}{k_2} + i\ \tfrac{2k_1 l\pi}{k_2}\right),$$
where $l = 0, 1, \ldots, k_2 - 1$. Thus t takes exactly k_2 different values. Since k_1 and k_2 are relatively prime numbers, $k_1 l/k_2$ is integer only if $l = 0$. Thus t^{k_1} takes exactly k_2 different values, i.e. $\gamma(t) = (z_1, x)$ has exactly k_2 solutions.
This proves that for each $(z_1, z_2) \in f^{-1}(c)$, there exists t such that $\gamma(t) = (z_1, z_2)$. Indeed, if it is not the case, then the number of solutions of $\gamma(t) = (z_1, x)$ is less than the number of solutions of $z_1^{k_1} x^{k_2} = c$. □

6.4.17 Proposition *$f_{n,q}$ is geometrically flat and the mapping*

$$Z_{f_{n,q}}\colon \mathbb{C} \longrightarrow Z_1(H(n,q))$$

is given by

$$Z_{f_{n,q}}(c) = \begin{cases} \left[f_{n,q}^{-1}(c)\right] & \text{if } c \neq 0 \\ \displaystyle\sum_{k=0}^{s+1} \nu_k[\sigma_k] & \text{if } c = 0. \end{cases}$$

Proof Since $f_{n,q}$ is surjective and open, $f_{n,q}$ is geometrically flat by lemma 2.5.4. Like in example 2.3.5,

$$Z_{f_{n,q}^{(k)}}(c) = \begin{cases} \left[(f_{n,q}^{(k)})^{-1}(c)\right] & \text{if } c \neq 0 \\ \nu_{k+1}[A_k] + \nu_k[B_k] & \text{if } c = 0, \end{cases}$$

where $A_k := \{v_k = 0\}$ and $B_k := \{u_k = 0\}$. Thus, if $c \neq 0$, then $Z_{f_{n,q}}(c) = \left[f_{n,q}^{-1}(c)\right]$, because $f_{n,q}^{-1}(c) = h_k((f_{n,q}^{(k)})^{-1}(c))$. Furthermore the set σ_k has really the multiplicity ν_k, by the definition of σ_k and the equation of $Z_{f_{n,q}^{(k)}}(c)$. $\square$

Let $\mathbb{F}_{n,q}$ be the coherent holomorphic foliation on $H(n,q)$ given by the mapping $f_{n,q}$. Since $f_{n,q}$ is simple, by [Rei97, Proposition 6.13], the fibres of $f_{n,q}$ are exactly the leaves of $\mathbb{F}_{n,q}$. Thus $\mathbb{F}_{n,q}$ is singular. We calculate in local coordinates that

$$\begin{aligned} \operatorname{Sing}\mathbb{F} &= \{h_k(0,0),\ 0 \leq k \leq s\} \\ &= \bigcup_{k=0}^{s} \sigma_k \cap \sigma_{k+1}. \end{aligned}$$

This foliation is generically regular, because

$$R^{\mathbb{F}}(\operatorname{Sing}\mathbb{F}) = f_{n,q}^{-1}(0) = \sigma \cup \sigma_0 \cup \sigma_{s+1},$$

which is a thin analytic subset of $H(n,q)$.

The mapping $Z_{f_{n,q}}$ induces a continuous mapping $\varphi\colon H(n,q) \longrightarrow Z_1(H(n,q))$ by $\varphi := Z_{f_{n,q}} \circ f_{n,q}$. It is an $\mathbb{F}$-invariant mapping and $|\varphi(x)| = f_{n,q}^{-1}(f_{n,q}(x)) = L_x$. By theorem 6.2.1, $H(n,q)/\mathbb{F}$ is a complex space. Finally, $H(n,q)/\mathbb{F} \cong \mathbb{C}$.

7 The meromorphic leaf space

A coherent holomorphic foliation $\mathbb{F}$ does not have leaves everywhere. In this case, the leaf space $X/\mathbb{F}$ is not defined. Even if it is defined, it is in general not a complex space. This section is dedicated to a generalisation of the leaf space, namely the meromorphic leaf space $Z(\mathbb{F})$. Under certain conditions, the meromorphic leaf space is well defined and even a complex space, even if $X/\mathbb{F}$ is not defined or not a complex space. The center of the proof is the theorem of Grauert-Siebert on meromorphic equivalence relations. (see theorem 2.6.8).

At the end of this section, we present some examples to better understand different cases that can occur.

In this subsection, $\mathbb{F}$ is an open singular holomorphic foliation of dimension d on a connected paracompact complex manifold X of dimension n. In general, $\mathbb{F}$ does not have leaves everywhere.

7.1 The subset $M^{\mathbb{F}}$ of $X \times X$

In section 6.2 we defined for an open coherent holomorphic foliation the sets $G(\mathbb{F})$ and $X_{\mathrm{tr}}(\mathbb{F})$ as well as the mapping $\zeta_{\mathbb{F}}\colon G(\mathbb{F}) \longrightarrow Z_d(X^{\mathrm{ns}})$. We define

$$C(\mathbb{F}) := \text{Interior of } \{x \in G(\mathbb{F}) \mid \zeta_{\mathbb{F}} \text{ is continuous in } x\}.$$

We write C if it is not necessary to precise the foliation. As in the regular case, one easily sees that C is open in X and $\mathbb{F}$-saturated.

7.1.1 Remark By theorem 6.1.1, the quotient space $C/\mathbb{F}^{\mathrm{ns}}$ is a complex space, and the canonical projection $\pi|_C\colon C \longrightarrow C/\mathbb{F}^{\mathrm{ns}}$ is open. Hence it is geometrically flat by lemma 2.5.4, since $C/\mathbb{F}^{\mathrm{ns}}$ is normal, and $\mathbb{F}|_C$ is stable in the sense of definition 5.1.4.

7.1.2 Definition An open coherent holomorphic foliation $\mathbb{F}$ is called ***generically stable*** if $X \setminus C$ is thin analytic in X.

7.1.3 Remark If $\mathbb{F}$ is generically stable, then it is generically regular, since $C \subset G \subset X^{\mathrm{ns}}$.

In the following, we suppose that $\mathbb{F}$ is generically stable.

By $R_C \subset C \times C$ we denote the analytic equivalence relation on C induced by $\pi|_C$, i.e. $R_C = R^{\mathbb{F}^{\mathrm{ns}}}|_C$. We define

$$M^{\mathbb{F}} := \overline{R_C} \subset X \times X.$$

By $p_j\colon M^{\mathbb{F}} \longrightarrow X$ we denote the projection on the jth factor.

7.1.4 Proposition *$M^{\mathbb{F}}$ is a relation*[9] *on X. In particular, $p_1(M^{\mathbb{F}}) = p_2(M^{\mathbb{F}}) = X$.*

Proof If $x \in X$, then there exists a sequence (x_k) in C such that $x_k \to x$. Thus $(x_k, x_k) \in R_C$ and $(x_k, x_k) \to (x, x)$. Hence $(x, x) \in M^{\mathbb{F}}$. For the symmetry, let $(x, y) \in M^{\mathbb{F}}$. Then there exists a sequence (x_k, y_k) in R_C such that $(x_k, y_k) \to (x, y)$. Thus $(y_k, x_k) \in R_C$ and $(y_k, x_k) \to (y, x)$ which proves that $(y, x) \in M^{\mathbb{F}}$. $\square$

7.1.5 Remark The equation $M^{\mathbb{F}} \cap (C \times C) = R_C$ holds. But $p_1^{-1}(x) \neq \{x\} \times L_x$ in general, even if $x \in C$, as it is shown by example 7.4.1 ($p_1 \colon M^{\mathbb{F}} \longrightarrow X$ is the projection on the first factor). The inclusion $p_1^{-1}(x) \supset \{x\} \times L_x$ holds: if $x \in C$, then $L_x = (\pi|_C)^{-1}(\pi|_C(x))$, because C is $\mathbb{F}$-saturated. Hence $\{x\} \times L_x = (R_C) \cap (\{x\} \times X) \subset M^{\mathbb{F}} \cap (\{x\} \times X) = p_1^{-1}(x)$.

7.1.6 Lemma *If $\mathbb{F}$ is a generically stable open foliation with leaves everywhere such that $X/\mathbb{F}$ is a complex space, then $M^{\mathbb{F}} = R^{\mathbb{F}}$.*

Proof C is $R^{\mathbb{F}}$-saturated and dense in X. Hence, claim 7.1.7 says that $R^{\mathbb{F}} = \overline{R_C} = M^{\mathbb{F}}$, which concludes the proof. $\square$

7.1.7 Claim *Let R be an open equivalence relation on X such that $R = \overline{R} \subset X \times X$. If U is an R-saturated dense open subset of X, then $R = \overline{R|_U}$.*

Proof Since $R|_U \subset R = \overline{R}$, the inclusion $\overline{R|_U} \subset R$ holds. We have to show that $R \subset \overline{R|_U}$.

Let $(x, y) \in R$. There exists a sequence (x_k) in U such that $x_k \to x$. Since R is open, there exists a sequence (y_k) in U such that $(x_k, y_k) \in R|_U$ and $y_k \to y$ (see lemma 1.1.2). Hence $(x_k, y_k) \to (x, y)$, which concludes the proof. $\square$

In general, $M^{\mathbb{F}}$ is not analytic, as it is shown by

7.1.8 Example Let

$$X := \{x \in \mathbb{C}^2 \mid |x_1| < 2,\ |x_2| < 3\} \smallsetminus \{x \in \mathbb{C}^2 \mid 1 \leq |x_1| < 2,\ 1 \leq |x_2| \leq 2\}$$

and let $\mathbb{F}$ be the foliation on X given by the mapping $f \colon X \longrightarrow \mathbb{C}$ with $f(x) := x_1$. Then

$$L_x = \begin{cases} \{y \in X \mid x_1 - y_1 = 0\} & \text{if } |x_1| < 1 \\ \{y \in X \mid x_1 - y_1 = 0,\ |y_2| < 1\} & \text{if } 1 \leq |x_1| < 2 \text{ and } |x_2| < 1 \\ \{y \in X \mid x_1 - y_1 = 0,\ 2 < |y_2| < 3\} & \text{if } 1 \leq |x_1| < 2 \text{ and } 2 < |x_2| < 3. \end{cases}$$

Figure 4 shows a representation of that foliation.

The leaf space $X/\mathbb{F}$ is not Hausdorff, and $C = X \smallsetminus \{x \in \mathbb{C}^2 \mid |x_1| = 1\}$, which is a dense subset of X. Hence

$$M^{\mathbb{F}} = R_C \cup \bigcup_{x \notin C} ((\{x\} \times f^{-1}(x)) \cup (f^{-1}(x) \times \{x\})).$$

[9] A **relation** R on X is a subset of $X \times X$ such that $(x, x) \in R$ for each $x \in X$ (reflexivity) and if $(x, y) \in R$, then $(y, x) \in R$ (symmetry).

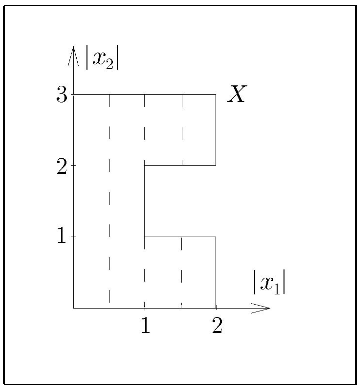

Figure 4: The foliation of example 7.1.8

We will prove that $M^{\mathbb{F}}$ is not analytic. Suppose the opposite. Let $x := (1, 1/2)$ and $x' := (1, 5/2)$. Then $(x, x') \in M^{\mathbb{F}}$, but $L_x \neq L_{x'}$. Let $0 < r < 1/2$ and let

$$\begin{aligned} U &:= \{y \in X \mid |y_1 - 1| < r,\ |y_2 - 1/2| < r\} \\ U' &:= \{y \in X \mid |y_1 - 1| < r,\ |y_2 - 5/2| < r\}. \end{aligned}$$

Since $M^{\mathbb{F}}$ is analytic by assumption, we can choose r such that there exists a mapping $g: U \times U' \longrightarrow \mathbb{C}$ with ($M^{\mathbb{F}}$ is pure 3-dimensional by proposition 7.1.12)

$$M^{\mathbb{F}} \cap (U \times U') = \{(y, y') \in U \times U' \mid g(y, y') = 0\}.$$

Let $\widetilde{U} := \{z \in \mathbb{C} \mid |z - 5/2| < r\}$ and let $\widetilde{g}: U \times \widetilde{U} \longrightarrow \mathbb{C}$ given by $\widetilde{g}(y_1, y_2, z) := g(y_1, y_2, y_1, z)$. If $(y_1, y_2, z) \in U \times \widetilde{U}$ such that $|y_1| < 1$, then $g(y_1, y_2, y_1, z) = 0$. Hence, by the Identity Theorem, $\widetilde{g} \equiv 0$. The point $(1 + r/2, 1/2, 1 + r/2, 5/2)$ is element of $U \times U'$, but not of $M^{\mathbb{F}}$. Furthermore, $g(1 + r/2, 1/2, 1 + r/2, 5/2) = \widetilde{g}(1 + r/2, 1/2, 5/2) = 0$, which is a contradiction. Hence $M^{\mathbb{F}}$ is not analytic.

7.1.9 Definition An open coherent holomorphic foliation $\mathbb{F}$ is called ***M*-analytic** if it is generically stable and if $M^{\mathbb{F}}$ is analytic in $X \times X$.

7.1.10 Remark If $X/\mathbb{F}$ is a complex space, then $M^{\mathbb{F}} = R^{\mathbb{F}}$ by lemma 7.1.6. Hence $\mathbb{F}$ is M-analytic.

7.1.11 Lemma *If $\mathbb{F}$ is M-analytic, then $M^{\mathbb{F}}$ is a meromorphic equivalence relation.*

Proof Let $P := X \setminus C$. According to the definition of C and the assumptions on $\mathbb{F}$, P is thin analytic in X. Finally $M^{\mathbb{F}} = \overline{R_C}$ is open by remark 7.1.1. $\square$

7.1.12 Proposition *If $\mathbb{F}$ is M-analytic, then $M^{\mathbb{F}}$ is pure $(d+n)$-dimensional, where $d = \dim \mathbb{F}$ and $n = \dim X$.*

Proof Let $(x,y) \in R_C$. Since $p_2|_{R_C} : R_C \longrightarrow C$ is open, the equation

$$\dim_{(x,y)} R_C = \dim_{(x,y)} (p_2|_{R_C})^{-1}(y) + \dim X$$

holds by [Fis76, 3.10]. Thus $\dim_{(x,y)} R_C = n + d$.
If $(x,y) \in M^{\mathbb{F}}$ such that $\dim_{(x,y)} M^{\mathbb{F}} \neq d+n$, then, by [KK83, 49.13], there exists $(x',y') \in M^{\mathbb{F}} \setminus \operatorname{Sing} M^{\mathbb{F}}$ such that $\dim_{(x',y')} M^{\mathbb{F}} \neq d+n$. Hence there exists an open neighbourhood U of (x',y') in $M^{\mathbb{F}} \setminus \operatorname{Sing} M^{\mathbb{F}}$ such that $\dim_u M^{\mathbb{F}} = \dim_{(x',y')} M^{\mathbb{F}} \neq d+n$ for each $u \in U$. Hence R_C is not dense in $M^{\mathbb{F}}$, which is a contradiction. $\square$

7.2 The meromorphic leaf space $Z(\mathbb{F})$ and the main theorem

We recall that the foliations we consider are open and in general do not have leaves everywhere.

7.2.1 Definition If $\mathbb{F}$ is an M-analytic singular holomorphic foliation, then we define

$$Z(\mathbb{F}) := \Phi_{M^{\mathbb{F}}} = p_{1*}(Z(p_2)) \subset Z_*(X).$$

$Z(\mathbb{F})$ is called the ***meromorphic leaf space*** of $\mathbb{F}$. An element $Z \in Z(\mathbb{F})$ is called a ***meromorphic leaf*** of $\mathbb{F}$.

7.2.2 Proposition *If $\mathbb{F}$ is an M-analytic singular holomorphic foliation with leaves everywhere such that $X/\mathbb{F}$ is a complex space, then $Z(\mathbb{F})$ is homeomorphic to $X/\mathbb{F}$.*

Proof By lemma 7.1.6, $R^{\mathbb{F}} = M^{\mathbb{F}}$. Furthermore, since p_2 is open and X is normal, $X_{\mathrm{gen}(p_2)} = X$ and $Z_{p_2} : X \longrightarrow Z_d(R^{\mathbb{F}})$ is well-defined and continuous. Similarly, $(X/\mathbb{F})_{\mathrm{gen}(\pi)} = X/\mathbb{F}$ and $Z_\pi : X/\mathbb{F} \longrightarrow Z_d(X)$ is well-defined and continuous. By [Sie94, Remark 5.2,5], $X/\mathbb{F}$ is homeomorphic to $Z_\pi(X/\mathbb{F})$. Applying lemma 2.3.8 at the commutative diagram

$$\begin{array}{ccc} R^{\mathbb{F}} & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & X/\mathbb{F}, \end{array}$$

the equation $Z_{p_2}(x) = Z_\pi(\pi(x)) \times [x]$ holds for each $x \in X$. Hence $Z(\mathbb{F}) = p_{1*}(Z(p_2)) = Z_\pi(X/\mathbb{F})$, which is homeomorphic to $X/\mathbb{F}$. $\square$

In general, $Z(\mathbb{F})$ is not homeomorphic to $X/\mathbb{F}$ as it is shown by example 7.4.3.

7.2.3 Proposition $\displaystyle\bigcup_{Z \in Z(\mathbb{F})} |Z| = X.$

Proof By proposition 2.4.6, $\bigcup_{Z \in Z(p_2)} |Z| = M^{\mathbb{F}}$. If $x \in X$, then $(x,x) \in M^{\mathbb{F}}$, and thus there exists $Z \in Z(p_2)$ such that $(x,x) \in |Z|$. Finally $x \in p_1(|Z|) = |p_{1*}(Z)|$ with $p_{1*}(Z) \in Z(\mathbb{F})$. $\square$

7.2.4 Definition An M-analytic singular holomorphic foliation $\mathbb{F}$ is called ***meromorphic leaf separable*** if for each $Z \in Z(\mathbb{F})$ there exist an open neighbourhood U of Z in $Z(\mathbb{F})$ and a relatively compact open subset B of X such that if $Z_1, Z_2 \in U$ with $Z_1|_B = Z_2|_B$, then $Z_1 = Z_2$.

Now we formulate the principal theorem of this subsection:

7.2.5 Theorem *Let $\mathbb{F}$ be an M-analytic singular holomorphic foliation. If $\mathbb{F}$ is meromorphic leaf separable then there exists a complex structure on $Z(\mathbb{F})$ with the following property: there exist a complex space X', a proper modification $\sigma\colon X' \longrightarrow X$ and a geometrically flat analytic equivalence relation R' on X' such that X'/R' is a complex space which is biholomorphic to $Z(\mathbb{F})$.*

7.2.6 Remarks

(a) If $X/\mathbb{F}$ is a complex space, then by proposition 7.2.2, $X = X'$ and $R' = R^{\mathbb{F}}$.

(b) The assumptions of theorem 7.2.5 may be satisfied, even if $X/\mathbb{F}$ is not a complex space or even if $\mathbb{F}$ does not have leaves everywhere (see examples of subsection 7.4). In this sense, $Z(\mathbb{F})$ is a generalisation of $X/\mathbb{F}$.

7.2.7 Corollary *If $\mathbb{F}$ is an M-analytic singular holomorphic foliation on a compact complex manifold X, then $Z(\mathbb{F})$ has a complex structure.*

Proof We have to show that $\mathbb{F}$ is meromorphic leaf separable. But, in the definition of meromorphic leaf separable, it suffices to take $U = Z(\mathbb{F})$ and $B = X$. $\square$

Now we prove theorem 7.2.5. First, we prove

7.2.8 Claim *If $\mathbb{F}$ is meromorphic leaf separable, then p_2 is fibre-cycle separable.*

Proof Using lemma 2.6.7, we identify $Z(p_2)$ with the space

$$\{(S, y) \in Z(\mathbb{F}) \times X \mid y \in |S|\}.$$

Let $Z_0 \in Z(p_2)$, i.e. $Z_0 = (S_0, y_0) \in Z(\mathbb{F}) \times X$ with $y_0 \in |S_0|$. By assumption, there exist an open neighbourhood U of S_0 in $Z(\mathbb{F})$ and a relatively compact open subset B of X such that if $S, S' \in U$ with $S|_B = S'|_B$, then $S = S'$. We choose a relatively compact open subset $V \Subset X$ of X such that $y_0 \in V$ and $V \cap p_2(p_1^{-1}(B)) \neq \emptyset$. This is possible, because if we choose a point $y \in p_2(p_1^{-1}(B))$, we find a relatively compact open subset V of X such that $y_0, y \in V$. Let

$$U' := (U \times V) \cap Z(p_2) \qquad B' := (B \times V) \cap M^{\mathbb{F}}.$$

The set U', resp. B', is open in $Z(p_2)$, resp in $M^{\mathbb{F}}$. Furthermore $U' \neq \emptyset$, because $Z_0 \in U'$, and $B' \neq \emptyset$, because $V \cap p_2(p_1^{-1}(B)) \neq \emptyset$. Finally B' is relatively compact in $M^{\mathbb{F}}$, because $M^{\mathbb{F}}$ is closed in $X \times X$ and $\overline{B \times V} = \overline{B} \times \overline{V}$ is compact in $X \times X$.

Let now $Z = (S, y) \in U'$ and $Z' = (S', y') \in U'$ such that $Z|_{B'} = Z'|_{B'}$. Thus $(S|_B, y) = (S'|_B, y')$, because $y, y' \in V$. Then $y = y'$ and $S = S'$. Hence p_2 is fibre-cycle separable. □

Proof of theorem 7.2.5 By claim 7.2.8 and theorem 2.4.8, $X' := (Z_{p_2}, {}_{Z_{p_2}}\mathcal{O})$ is a complex space. Hence the conditions of theorem 2.6.8 are satisfied for $M^{\mathbb{F}}$ by claim 7.2.8. Thus the conclusions of theorem 7.2.5 follow immediately from theorem 2.6.8. □

7.2.9 Remark In general the equivalence classes of R' are not connected as it is shown by example 7.4.3. Hence, in general, there does not exist a foliation $\mathbb{F}'$ on X' such that $R' = R^{\mathbb{F}'}$.

But, we can prove the following:

7.2.10 Proposition *In theorem 7.2.5, if X' is a manifold and R' is simple*[10] *then there exists a foliation $\mathbb{F}'$ on X' such that $R^{\mathbb{F}'} = R'$.*

Proof The canonical projection $q : X' \longrightarrow X'/R'$ is open and simple [11]. Let $\mathbb{F}'$ be the foliation given by q. By [Rei97, Proposition 6.13] and [Rei97, Theorem 6.26], $R^{\mathbb{F}'} = R_q = R'$ (where R_q is the equivalence relation defined by q). □

7.3 Description of $Z(\mathbb{F})$ in particular cases

The construction of $Z(\mathbb{F})$ is abstract. We will present here two descriptions of this space in two particular cases. The first case is if all leaves in C are closed in X. The second is if the foliation is given by a simple global integral.

We define

$$X^{\mathrm{cl}}(\mathbb{F}) := \{x \in X^{\mathrm{ns}}(\mathbb{F}) \mid L_x \text{ is closed in } X\}.$$

We write X^{cl} if it is not necessary to precise the foliation. This set is $\mathbb{F}$-saturated, but it could be empty, as it is shown by example 7.4.1.

7.3.1 Motivation For $x \in X^{\mathrm{cl}}$, L_x is an analytic subset of X^{ns} and of X. Hence, if $x \in G \cap X^{\mathrm{cl}}$, then the cycle $\zeta_{\mathbb{F}}(x) \in Z_d(X^{\mathrm{ns}})$ can be interpreted as a cycle in $Z_d(X)$.

According to the previous motivation, we define a mapping

$$\zeta_{\mathbb{F}}^X : G \cap X^{\mathrm{cl}} \longrightarrow Z_d(X), \quad \zeta_{\mathbb{F}}^X(x) := \mu_t(L_x)[L_x].$$

We define

$$C^{\mathrm{cl}}(\mathbb{F}) := \text{Interior of } \{x \in G \cap X^{\mathrm{cl}} \mid \zeta_{\mathbb{F}}^X \text{ is continuous at } x\}.$$

[10] An equivalence relation R is called **simple** if for each $x \in X$, the equivalence class $R(x)$ is connected.

[11] A mapping $f : X \longrightarrow Y$ between topological spaces is called **simple** if each fibre of f is connected in X

We write C^{cl} if it is not necessary to precise the foliation. The inclusion $C^{\mathrm{cl}} \subset C$ holds.

7.3.2 Theorem *If $\mathbb{F}$ is an M-analytic singular holomorphic foliation and if C^{cl} is a dense subset of X, then $Z(\mathbb{F}) = \overline{\zeta_{\mathbb{F}}^{X}(C^{\mathrm{cl}})} \setminus \{[\emptyset]\}$.*

7.3.3 Definition If $\mathbb{F}$ is an open singular holomorphic foliation on X such that C^{cl} is dense in X, then we define

$$Z(\mathbb{F}) := \overline{\zeta_{\mathbb{F}}^{X}(C^{\mathrm{cl}})} \setminus \{[\emptyset]\}.$$

An example of this definition is given by example 7.4.4.

7.3.4 Remark If $\mathbb{F}$ is an open M-analytic singular holomorphic foliation such that C^{cl} is dense in X, we can define $Z(\mathbb{F})$ in two different way (definition 7.2.1 or definition 7.3.3). But, these two definitions coincide by theorem 7.3.2.

7.3.5 Remark We have to drop $[\emptyset]$ in definition 7.3.3, because if C^{cl} is not relatively compact in X, then $[\emptyset] \in \overline{\zeta_{\mathbb{F}}^{X}(C^{\mathrm{cl}})}$ (compare remark 2.4.2).

On the other hand, we prove also the following

7.3.6 Theorem *Let $\mathbb{F}$ be an M-analytic singular holomorphic foliation on X with a global simple integral $f\colon X \longrightarrow Y$. If $f^{-1}\left(\overline{f(\operatorname{Sg} f)}\right)$ is nowhere dense in X and if $f(X) \setminus \overline{f(\operatorname{Sg} f)}$ is dense in $f(X)$, then $Z(\mathbb{F}) = Z(f)$.*

Many examples of foliations are constructed with a global integral. This theorem helps us to construct examples for theorem 7.2.5.

For the proof of theorem 7.3.2, we use the following

7.3.7 Claim *Under the assumptions of theorem 7.3.2, $M^{\mathbb{F}} = \overline{R_{C^{\mathrm{cl}}}}$ (where $R_{C^{\mathrm{cl}}} = (R_C)|_{C^{\mathrm{cl}}}$).*

Proof It suffices to prove that $R_C \subset \overline{R_{C^{\mathrm{cl}}}}$. Let $(x, y) \in R_C$. Since C^{cl} is dense in X, there exists a sequence (x_k) in C^{cl} such that $x_k \to x$. By continuity of $(\zeta_{\mathbb{F}})|_C\colon C \longrightarrow Z_d(X^{\mathrm{ns}})$, we obtain $\zeta_{\mathbb{F}}(x_k) \to \zeta_{\mathbb{F}}(x) = \zeta_{\mathbb{F}}(y)$. Hence (L_{x_k}) converges set theoretically to L_y as closed subset of X^{ns}, and thus there exists, for each k, $y_k \in L_{x_k} \subset C^{\mathrm{cl}}$ such that a subsequence (y_{k_l}) converges to y. Thus $(x_{k_l}, y_{k_l}) \to (x, y)$. □

7.3.8 Claim *Under the assumptions of theorem 7.3.2, for each $x \in C^{\mathrm{cl}}$, $p_1^{-1}(x) = \{x\} \times L_x$ (where $p_1\colon M^{\mathbb{F}} \longrightarrow X$ is the projection on the first factor).*

Proof Let $x \in C^{\mathrm{cl}}$. By remark 7.1.5, we have to prove that $p_1^{-1}(x) \subset \{x\} \times L_x$. Let $(x, y) \in p_1^{-1}(x) \subset M^{\mathbb{F}}$. By claim 7.3.7, $M^{\mathbb{F}} = \overline{R_{C^{\mathrm{cl}}}}$. Thus there exists a sequence (x_k, y_k) in $R_{C^{\mathrm{cl}}}$ such that $(x_k, y_k) \to (x, y)$. Furthermore $\zeta_{\mathbb{F}}^X(y_k) = \zeta_{\mathbb{F}}^X(x_k) \to \zeta_{\mathbb{F}}^X(x) \in Z_d(X)$. Thus $L_{y_k} \to L_x$, i.e.

$$L_x = \{\eta \in X \mid \eta \text{ is an accumulation point of a sequence } (\eta_k) \text{ such that } \eta_k \in L_{x_k} \text{ for each } k\}.$$

Hence $y \in L_x$. □

7.3.9 Claim *Under the assumptions of theorem 7.3.2, $C^{\mathrm{cl}} \subset X_{\mathrm{gen}(p_2)}$ (where the mapping $p_2 : M^{\mathbb{F}} \longrightarrow X$ is the projection on the second factor).*

Proof Let

$$E_{p_2} := \{(x, y) \in M^{\mathbb{F}} \mid \dim_{(x,y)} p_2^{-1}(y) > \dim_{(x,y)} M^{\mathbb{F}} - \dim_y X\}.$$

If $y \in C^{\mathrm{cl}}$, then $p_2^{-1}(y) = L_y \times \{y\}$ by claim 7.3.8. Hence $p_2^{-1}(y)$ has pure dimension $d = \dim M^{\mathbb{F}} - \dim X$. Hence $y \notin p_2(E_{p_2})$. Thus

$$C^{\mathrm{cl}} \subset X \setminus p_2(E_{p_2}) = p_2(M^{\mathbb{F}}) \setminus p_2(E_{p_2}) = X_{\mathrm{gen}(p_2)},$$

which completes the proof. □

Proof of theorem 7.3.2 If $y \in C^{\mathrm{cl}}$, then $y \in X_{\mathrm{gen}(p_2)}$ by claim 7.3.9. Hence $Z_{p_2,\mathrm{gen}}(y) \in Z_d(M^{\mathbb{F}})$ is well-defined. By claim 7.3.8, $|Z_{p_2,\mathrm{gen}}(y)| = p_2^{-1}(y) = L_y \times \{y\}$, which is an analytic subset of $R_{C^{\mathrm{cl}}}$. Hence $Z_{p_2,\mathrm{gen}}(y)$ can be interpreted as a cycle in $Z_d(R_{C^{\mathrm{cl}}})$. By $\tau := \pi|_{C^{\mathrm{cl}}} : C^{\mathrm{cl}} \longrightarrow C^{\mathrm{cl}}/\mathbb{F}|_{C^{\mathrm{cl}}}$ we denote the canonical projection. Then, applying lemma 2.3.8 at the commutative diagram

$$\begin{array}{ccc} R_{C^{\mathrm{cl}}} & \xrightarrow{p_2} & C^{\mathrm{cl}} \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle \tau} \\ C^{\mathrm{cl}} & \xrightarrow{\tau} & C^{\mathrm{cl}}/\mathbb{F}, \end{array}$$

the equation

$$Z_{p_2,\mathrm{gen}}(y) = Z_{\tau,\mathrm{gen}}(\tau(y)) \times [y] \in Z_d(R_{C^{\mathrm{cl}}})$$

holds. Note that $Z_{\tau,\mathrm{gen}}(\tau(y)) \in Z_d(C^{\mathrm{cl}})$. Since $|Z_{\tau,\mathrm{gen}}(\tau(y))| = L_x$ is an analytic subset of X, $Z_{\tau,\mathrm{gen}}(\tau(y))$ can be interpreted as a cycle in $Z_d(X)$. Since $Z_{\tau,\mathrm{gen}}(\tau(y)) = \mu_a(L_y)[L_y]$, the equation $Z_{\tau,\mathrm{gen}}(\tau(y)) = \zeta_{\mathbb{F}}^X(y)$ holds by theorem 5.3.4. Thus

$$p_{1*}(Z_{p_2,\mathrm{gen}}(y)) = Z_{\tau,\mathrm{gen}}(\tau(y)) = \zeta_{\mathbb{F}}^X(y) \in Z_d(X).$$

Since C^{cl} is dense in $X_{\mathrm{gen}(p_2)}$ and $p_{1*} \circ Z_{p_2,\mathrm{gen}} : X_{\mathrm{gen}(p_2)} \longrightarrow Z_d(X)$ is continuous, the equation

$$\overline{p_{1*}(Z_{p_2,\mathrm{gen}}(C^{\mathrm{cl}}))} = \overline{p_{1*}(Z_{p_2,\mathrm{gen}}(X_{\mathrm{gen}(p_2)}))}$$

holds. Furthermore, by remark 2.6.6, the equation

$$p_{1*}(\overline{Z_{p_2,\mathrm{gen}}(X_{\mathrm{gen}(p_2)})} \setminus \{[\emptyset]\}) = \overline{p_{1*}(Z_{p_2,\mathrm{gen}}(X_{\mathrm{gen}(p_2)}))} \setminus \{[\emptyset]\}$$

holds. Thus,

$$\begin{aligned} Z(\mathbb{F}) &= p_{1*}(\overline{Z_{p2,\mathrm{gen}}(X_{\mathrm{gen}(p2)})} \smallsetminus \{[\emptyset]\}) \\ &= \overline{p_{1*}(Z_{p2,\mathrm{gen}}(X_{\mathrm{gen}(p2)}))} \smallsetminus \{[\emptyset]\} \\ &= \overline{p_{1*}(Z_{p2,\mathrm{gen}}(C^{\mathrm{cl}}))} \smallsetminus \{[\emptyset]\} \\ &= \overline{\zeta_{\mathbb{F}}^{X}(C^{\mathrm{cl}})} \smallsetminus \{[\emptyset]\}, \end{aligned}$$

which completes the proof. □

For the proof of theorem 7.3.6, we need different technical claims.

7.3.10 Claim *Under the assumptions of theorem 7.3.6,* $R^{\mathbb{F}}(\operatorname{Sg} f) \subset f^{-1}(f(\operatorname{Sg} f))$.

Proof Let $x \in R^{\mathbb{F}}(\operatorname{Sg} f)$, i.e. there exists $y \in \operatorname{Sg} f \cap L_x$. By lemma 3.3.3, $L_x = L_y \subset f^{-1}(f(y))$. Thus $x \in f^{-1}(f(\operatorname{Sg} f))$, which proves the inclusion. □

7.3.11 Claim *Under the assumptions of theorem 7.3.6, if* $x \in X \smallsetminus f^{-1}(f(\operatorname{Sg} f))$, *then* $L_x \cap \operatorname{Sing} \mathbb{F} = \emptyset$ *and* $L_x = f^{-1}(f(x))$.

Proof By claim 7.3.10, if $x \notin f^{-1}(f(\operatorname{Sg} f))$, then $x \notin R^{\mathbb{F}}(\operatorname{Sg} f)$. Thus $L_x \cap \operatorname{Sg} f = \emptyset$ and hence $L_x \cap \operatorname{Sing} \mathbb{F} = \emptyset$ (because $\operatorname{Sing} \mathbb{F} \subset \operatorname{Sg} f$). Furthermore $f^{-1}(f(x)) \subset X \smallsetminus \operatorname{Sg} f$: if $y \in f^{-1}(f(x))$, then $f(y) = f(x) \notin f(\operatorname{Sg} f)$ and hence $y \notin \operatorname{Sg} f$. Since $L_x \subset f^{-1}(f(x)) \subset X \smallsetminus \operatorname{Sg} f$ and $f^{-1}(f(x))$ is connected and $f|_{X \smallsetminus \operatorname{Sg} f}$ is a local regular foliation of $\mathbb{F}^{\mathrm{reg}}$, $f^{-1}(f(x)) = L_x$. □

7.3.12 Claim *Under the assumptions of theorem 7.3.6,* $f(X) \smallsetminus f(\operatorname{Sg} f) \subset Y_{\mathrm{gen}(f)}$.

Proof In definition 2.3.3 we defined $Y_{\mathrm{gen}(f)} = f(X) \smallsetminus (f(E) \cup N)$. Hence we have to prove that $(f(E) \cup N) \cap f(X) \subset f(\operatorname{Sg} f)$. Since $N \cap f(X) \subset \operatorname{Sing} Y \subset f(\operatorname{Sg} f)$, the inclusion $E \subset \operatorname{Sg} f$ completes the proof. For this last inclusion, let $x \notin \operatorname{Sg} f$. Since $f|_{X \smallsetminus \operatorname{Sg} f}$ is open and $f(X \smallsetminus \operatorname{Sg} f) \subset Y \smallsetminus \operatorname{Sing} Y$, the equation

$$\dim_x f^{-1}(f(x)) = \dim_x X - \dim_{f(x)} Y$$

holds, i.e. $x \notin E$. □

7.3.13 Claim *Under the assumptions of theorem 7.3.6, for each* $x \notin f^{-1}(\overline{f(\operatorname{Sg} f)})$, $Z_{f,\mathrm{gen}}(f(x)) = \mu_t(L_x) L_x$. *Furthermore,* $X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)}) \subset C^{\mathrm{cl}}$.

Proof By claim 7.3.12, $f(X) \smallsetminus \overline{f(\operatorname{Sg} f)} \subset f(X) \smallsetminus f(\operatorname{Sg} f) \subset Y_{\mathrm{gen}(f)}$. Hence, if $x \notin f^{-1}(\overline{f(\operatorname{Sg} f)})$, then $f(x) \in Y_{\mathrm{gen}(f)}$ and $Z_{f,\mathrm{gen}}(f(x))$ is well defined. By claim 7.3.11, $f^{-1}(f(x)) = L_x$. Hence, to prove that $Z_{f,\mathrm{gen}}(f(x)) = \mu_t(L_x) L_x$, we have only to control the multiplicity. By claim 7.3.11, $L_x \cap \operatorname{Sing} \mathbb{F} = \emptyset$. Let (U, p) be a local $\mathbb{F}^{\mathrm{reg}}$-foliation at x, with $U \subset X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})$, such that $\mu_t(L_x) = \max_{v \in p(U)} (\nu_p(v))$ (see lemma 5.2.11). Then

$$\begin{aligned} (f,q)^{-1}((f,q)(y)) &= \{y' \in U \mid y' \in f^{-1}(f(y)),\ q(y) = q(y')\} \\ &= \{y' \in U \mid y' \in L_y,\ q(y) = q(y')\}. \end{aligned}$$

Thus the number of elements of $(f,q)^{-1}((f,q)(y))$ is exactly the number of elements of $p(L_y)$. Hence $\mu_t(L_x)$ is the number of sheets of (f,q).
By claim 7.3.11, if $X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})$, then $f^{-1}(f(x)) = L_x$, and hence L_x is closed in X. Furthermore, the topological multiplicity of L is well-defined for each $L \subset X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})$. Hence $X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)}) \subset G \cap X^{\mathrm{cl}}$. The facts that $Z_{f,\mathrm{gen}}(x) = \zeta_{\mathbb{F}}^{X}(x)$ for each $x \in X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})$ and that $Z_{f,\mathrm{gen}}$ is continuous prove that $X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)}) \subset C^{\mathrm{cl}}$. □

Proof of theorem 7.3.6 By claim 7.3.12, $f(X) \smallsetminus \overline{f(\operatorname{Sg} f)} \subset f(X) \smallsetminus f(\operatorname{Sg} f) \subset Y_{\mathrm{gen}(f)}$. By claim 7.3.13,

$$\zeta_{\mathbb{F}}^{X}\left(X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})\right) = Z_{f,\mathrm{gen}}\left(f(X) \smallsetminus \overline{f(\operatorname{Sg} f)}\right).$$

To complete the proof, we have to show that

(a) $Z(\mathbb{F}) = \overline{\zeta_{\mathbb{F}}^{X}\left(X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})\right)} \smallsetminus \{[\emptyset]\}$;

(b) $Z(f) = \overline{Z_{f,\mathrm{gen}}\left(f(X) \smallsetminus \overline{f(\operatorname{Sg} f)}\right)} \smallsetminus \{[\emptyset]\}$.

Since $f^{-1}(\overline{f(\operatorname{Sg} f)})$ is nowhere dense in X, $X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})$ is dense in X. Thus, by claim 7.3.13, C^{cl} is dense in X. Hence $Z(\mathbb{F}) = \overline{\zeta_{\mathbb{F}}^{X}(C^{\mathrm{cl}})} \smallsetminus \{[\emptyset]\} \subset Z_d(X)$ by theorem 7.3.2. Since $X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})$ is dense in C^{cl} and by the continuity of $\zeta_{\mathbb{F}}^{X}$,

$$Z(\mathbb{F}) = \overline{\zeta_{\mathbb{F}}^{X}(C^{\mathrm{cl}})} \smallsetminus \{[\emptyset]\} = \overline{\zeta_{\mathbb{F}}^{X}\left(X \smallsetminus f^{-1}(\overline{f(\operatorname{Sg} f)})\right)} \smallsetminus \{[\emptyset]\},$$

which completes the proof of a). By the assumption that $f(X) \smallsetminus \overline{f(\operatorname{Sg} f)}$ is dense in $f(X)$ and by the continuity of $Z_{f,\mathrm{gen}}$,

$$Z(f) = \overline{Z_{f,\mathrm{gen}}(Y_{\mathrm{gen}(f)})} \smallsetminus \{[\emptyset]\} = \overline{Z_{f,\mathrm{gen}}\left(f(X) \smallsetminus \overline{f(\operatorname{Sg} f)}\right)} \smallsetminus \{[\emptyset]\},$$

which completes the proof of b). □

7.4 Examples

In this subsection, we present some examples of the previously developed theory. We adopt the notations of the subsections 7.1, 7.2 and 7.3.

We begin with a classical simple example.

7.4.1 Example Consider the foliation $\mathbb{F}$ on $X := \mathbb{C}^2$ given by the following $\mathbb{C}^*$-action on X:

$$\begin{array}{rcl} \mathbb{C}^* \times X & \to & X \\ (\lambda, x) & \mapsto & \lambda x \end{array}$$

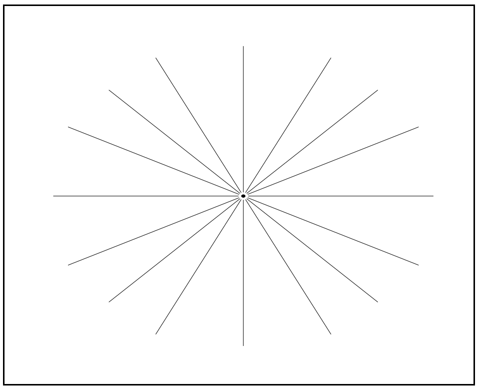

Figure 5: The foliation of example 7.4.1

Denote $X^* := (\mathbb{C}^2)^*$. The foliation $\mathbb{F}$ does not have leaves everywhere: $\Sigma(\mathbb{F}) = \{0\}$. For $x \in X^\rho = X^*$, the leaf L_x through x is given by

$$L_x = \{\lambda x \in X^* \mid \lambda \in \mathbb{C}^*\}.$$

Figure 5 shows the leaves of that foliation.

Since L_x is not closed in X for each $x \in X^*$, $X^{\mathrm{cl}} = \emptyset$. Clearly

$$\{L_x \mid x \in X^*\} = X^*/\mathbb{F}|_{X^*} \cong \mathbb{P}_1.$$

Since $X^*/\mathbb{F}$ is a complex space, $G = C = X^*$. One sees that $\zeta_{\mathbb{F}}\colon C \longrightarrow Z_1(X^*)$ is given by $\zeta_{\mathbb{F}}(x) = [L_x]$. Thus

$$R_C = \{(x,y) \in X^* \times X^* \mid x_1 y_2 = x_2 y_1\},$$

and

$$M^{\mathbb{F}} = \overline{R_C} = \{(x,y) \in X \times X \mid x_1 y_2 = x_2 y_1\}.$$

This is an analytic subset of $X \times X$. Denote by $p_j\colon M^{\mathbb{F}} \longrightarrow X$ the projection on the jth factor. Then, for each $x \neq (0,0)$,

$$\begin{array}{rcl} \{x\} \times L_x & = & \{x\} \times \{y \in X^* \mid x_2 y_1 = y_2 x_1\} \cong \mathbb{C}^* \quad \text{and} \\ p_1^{-1}(x) & = & \{x\} \times \{y \in X \mid x_2 y_1 = y_2 x_1\} \cong \mathbb{C}. \end{array}$$

Hence $\{x\} \times L_x \subsetneq p_1^{-1}(x)$ (see remark 7.1.5).
We have to find $Z(p_2)$. The set $M^{\mathbb{F}}$ is 3-dimensional by proposition 7.1.12. Hence

$$E_{p_2} = \{(x,y) \in M^{\mathbb{F}} \mid \dim_{(x,y)} p_2^{-1}(y) > 1\} = \mathbb{C}^2 \times \{0\}.$$

Thus, $p_2(E_{p_2}) = \{0\}$ and finally $X_{\mathrm{gen}(p_2)} = X^*$. We can see that for $y \in X^*$,

$$Z_{p_2,\mathrm{gen}}(y) = [\{\lambda y \mid \lambda \in \mathbb{C}\} \times \{y\}].$$

One sees that

$$Z(p_2) = \overline{Z_{p_2,\mathrm{gen}}(X^*)} \smallsetminus \{[\emptyset]\} = Z_{p_2,\mathrm{gen}}(X^*) \cup \left\{ [\{\lambda x \mid \lambda \in \mathbb{C}\} \times \{0\}] \,\middle|\, x \in X^* \right\}.$$

Thus

$$Z(\mathbb{F}) = p_{1*}(Z(p_2)) = \left\{ [\{\lambda x \mid \lambda \in \mathbb{C}\}] \,\middle|\, x \in X^* \right\} \cong X^*/\mathbb{F} \cong \mathbb{P}_1.$$

Let $\widetilde{\mathbb{C}^2} := \{(Z, [\zeta] \in \mathbb{C}^2 \times \mathbb{P}_1 \mid z_1\zeta_2 = z_2\zeta_1\}$ be the blowing-up of $\mathbb{C}^2$ in the point 0 (see for example [KK83, §32B] or [BK81, §8.4]). Hence, using the characterisation of $X' := Z(p_2)$ of lemma 2.6.7, X' is isomorphic to $\widetilde{\mathbb{C}^2}$. Furthermore $\sigma : X' \longrightarrow X$ is the blowing-down mapping

The following example is also classical. It is an application of theorem 7.3.6.

7.4.2 Example Consider the foliation $\mathbb{F}$ on $X = \mathbb{C}^3$ given by the following $\mathbb{C}^*$-action on X:

$$\begin{array}{ccc} \mathbb{C}^* \times X & \to & X \\ (\lambda, x) & \mapsto & (\lambda x_1, \lambda x_2, \lambda^{-1} x_3) \end{array}$$

The foliation has not leaves everywhere and $\mathrm{Sh}\mathbb{F} = \Sigma(\mathbb{F}) = \{0\}$. For $x \in X^\rho = X^{\mathrm{ns}} = X \smallsetminus \{0\}$, the leaf L_x through x is given by

$$L_x = \{(\lambda x_1, \lambda x_2, \lambda^{-1} x_3) \in X^\rho \mid \lambda \in \mathbb{C}^*\}.$$

Figure 6 shows the leaves of that foliation.

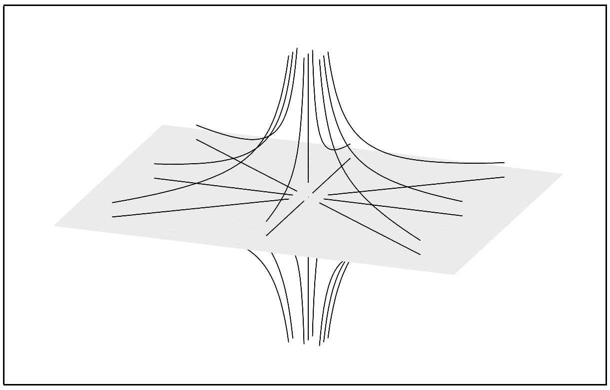

Figure 6: The foliation of example 7.4.2

One sees that $\zeta_{\mathbb{F}} : X^{\mathrm{ns}} \longrightarrow Z_1(X^{\mathrm{ns}})$ is given by $\zeta_{\mathbb{F}}(x) = [L_x]$. Thus

$$C = X \smallsetminus \{x \in \mathbb{C}^3 \mid x_1x_3 = x_2x_3 = 0\}.$$

The foliation has a global integral, namely the mapping $f : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ given by $f(x) = (x_1x_3, x_2x_3)$. This mapping is simple. One sees that $\mathrm{Sg}\, f = \{x_3 = 0\}$, $f^{-1}(f(\mathrm{Sg}\, f)) = \{x_1x_3 = x_2x_3 = 0\}$ and $f(\mathrm{Sg}\, f) = \{0\}$. Hence the assumptions of theorem 7.3.6 are satisfied. Hence $Z(\mathbb{F}) = Z(f)$. For each $x \in C$, $L_x = f^{-1}(f(x))$. Thus by definition of $Z(f)$ and by the fact that $f(C) = \mathbb{C}^2 \setminus \{0\}$ is dense in $\mathbb{C}^2$,

$$\begin{aligned} Z(f) &= \overline{Z_{f,\mathrm{gen}}(C)} \setminus \{[\emptyset]\} = \overline{\Big\{[f^{-1}(c)] \,\Big|\, c \in \mathbb{C}^2 \setminus \{0\}\Big\}} \setminus \{[\emptyset]\} \\ &= \Big\{[f^{-1}(c)] \,\Big|\, c \in \mathbb{C}^2 \setminus \{0\}\Big\} \cup \Big\{[H_\zeta] + [V] \,\Big|\, \zeta \in \mathbb{P}_1\Big\}, \end{aligned}$$

where $H_\zeta := \{(\lambda\zeta_1, \lambda\zeta_2, 0) \in \mathbb{C}^3 \mid \lambda \in \mathbb{C}\}$ and $V = \{(0,0,\lambda) \mid \lambda \in \mathbb{C}\}$. Let

$$\widetilde{\mathbb{C}^2} := \{(Z, [\zeta] \in \mathbb{C}^2 \times \mathbb{P}_1 \mid z_1\zeta_2 = z_2\zeta_1\}$$

be the blowing-up of $\mathbb{C}^2$ in the point 0. Then there exists a bijection $\psi\colon \widetilde{\mathbb{C}^2} \longrightarrow Z(f)$ given by

$$\psi(z, [\zeta]) = \begin{cases} [f^{-1}(z)] & \text{if } z \neq 0 \\ [H_\zeta] + [V] & \text{if } z = 0. \end{cases}$$

Doing the same argumentation as in example 2.6.4,

$$\begin{aligned} R_C &= \{(x,y) \in C \times C \mid x_1x_3 = y_1y_3,\ x_2x_3 = y_2y_3\}; \\ M^{\mathbb{F}} &= \overline{R_C} = \{(x,y) \in C \times C \mid x_1x_3 = y_1y_3,\ x_2x_3 = y_2y_3,\ x_1y_2 = x_2y_1\}. \end{aligned}$$

By lemma 2.6.7,

$$\begin{aligned} Z(p_2) &= \{(S,x) \in Z(\mathbb{F}) \times X \mid x \in |S|\} \\ &\cong \{(z,[\zeta],x) \in \widetilde{\mathbb{C}^2} \times X \mid x \in |\psi(x,[\zeta])|\} \\ &\cong \{(z,[\zeta],x) \in \widetilde{\mathbb{C}^2} \times X \mid f(x) = z,\ \zeta_1x_2 = x_1\zeta_2\}. \end{aligned}$$

By the description of R' in theorem 2.6.8, we conclude that

$$R' = \Big\{((z,[\zeta],x),(z',[\zeta'],y)) \in Z(p_2) \times Z(p_2) \,\Big|\, (z,[\zeta]) = (z',[\zeta'])\Big\}.$$

The next example illustrates remark 7.2.9:

7.4.3 Example Let $\mathbb{F}$ be the regular foliation on $X := \mathbb{C}^2 \setminus \{0\}$ given by the submersion $f\colon X \longrightarrow \mathbb{C}$, with $f(x) = x_1x_2$. Denoting $A := \{x \in X \mid x_2 = 0\}$ and $B := \{x \in X \mid x_1 = 0\}$,

$$L_x = \begin{cases} f^{-1}(f(x)) & \text{if } x_1x_2 \neq 0 \\ A & \text{if } x \in A \\ B & \text{if } x \in B \end{cases}$$

Figure 7 shows the leaves of that foliation.

Since $\mathbb{F}$ is regular, $X^{\mathrm{ns}} = X$. Furthermore, $G = X$ and $C = X \setminus \{x \in X \mid x_1x_2 = 0\}$. One sees that

$$M^{\mathbb{F}} = \overline{R_C} = \{(x,y) \in X \times X \mid x_1x_2 = y_1y_2\}.$$

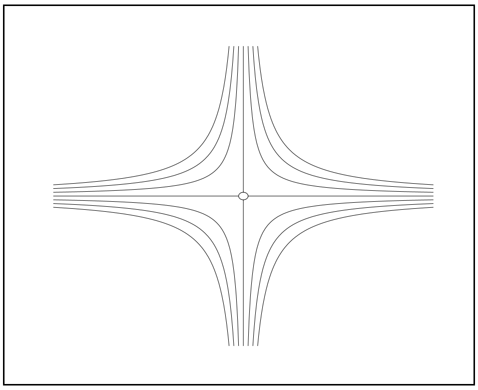

Figure 7: The foliation of example 7.4.3

This is an open analytic equivalence relation on X. By theorem 7.3.2

$$\begin{array}{rcl} Z(\mathbb{F}) & = & \overline{\zeta_{\mathbb{F}}(C)} \setminus \{[\emptyset]\} \\ & = & \{[L_x] \mid x \in C\} \cup \{[A]+[B]\} \\ & \cong & \mathbb{C}. \end{array}$$

Since $p_2 : M^{\mathbb{F}} \longrightarrow X$ is open and X is a manifold, p_2 is geometrically flat and thus $Z(p_2) \cong X$. Hence $R' = M^{\mathbb{F}}$. There does not exist a foliation $\mathbb{F}'$ on X such that $M^{\mathbb{F}} = R^{\mathbb{F}'}$, because the class $R'(1,0) = A \cup B$ is not connected.

7.4.4 Example Let X and $\mathbb{F}$ be as in example 7.1.8. One sees that

$$C = C^{\mathrm{cl}} = \{x \in X \mid |x_1| \neq 1\}.$$

We have seen that $M^{\mathbb{F}}$ is not analytic. Then, by definition 7.3.3

$$\begin{array}{rcl} Z(\mathbb{F}) & = & \overline{\zeta^X_{\mathbb{F}}(C^{\mathrm{cl}})} \setminus \{[\emptyset]\} \\ & = & \zeta^X_{\mathbb{F}}(C^{\mathrm{cl}}) \cup \{[L_{(z,1/2)}] \mid |z| = 1\} \cup \{[L_{(z,3/2)}] \mid |z| = 1\} \cup \\ & & \cup \{[L_{(z,1/2)}] + [L_{z,3/2)}] \mid |z| = 1\}. \end{array}$$

Furthermore, $Z(\mathbb{F})$ is meromorphic leaf separable.

List of examples

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Glossary of notations

$A \amalg B$	Coproduct (or disjoint union) of the two sets A and B
$U \Subset X$	U is open in X
${}_X\mathcal{C}$, $\mathcal{C}$	Sheaf of continuous functions on X (p. **7**)
${}_X\mathcal{O}$, $\mathcal{O}$	Structure sheaf of the complex space X (p. **7**)
$\mathcal{O}(U)$ $(\mathcal{C}(U))$	Holomorphic (continuous) functions on U (p. **7**)
$\widetilde{\mathcal{O}}$	Sheaf of weakly holomorphic functions (p. **9**)
$\widehat{\mathcal{O}}$	$\widetilde{\mathcal{O}} \cap \mathcal{C}$ (p. **9**)
A_x	Germ of the subset A at x (p. **7**)
$R(x)$	Equivalence class of x (p. **8**)
$R(A)$	R-saturated hull of A (p. **8**)
$R\|_A$	Restriction of R on A (p. **8**)
X/R	Quotient of X by R (p. **8**)
$\widetilde{X}$	Normalization of X (p. **9**)
$\widehat{X}$	Maximalization of X (p. **9**)
$\operatorname{Sing} f$	Singular locus of the holomorphic mapping f (p. **10**)
$\operatorname{Sg} f$	Singular set of the generically open mapping f (p. **10**)
$[\emptyset]$	Null cycle (p. **13**)
$\|Z\|$	Support of the cycle Z (p. **13**)
$[S]$	Reduced cycle composed of irreducible components of the analytic set S (p. **13**)
$\mathcal{S} := (\chi : V \longrightarrow \Omega, W, D)$	Scale (p. **17**)
$\deg_{\mathcal{S}}(Z)$	Degree of a cycle Z relatively to a scale $\mathcal{S}$ (p. **15**)

$Z_d(X)$	Set of pure d-dimensional cycles (p. **13**)
$Z_*(X)$	Set of all analytic cycles (p. **17**)
$C_d(X)$	set of d-dimensional compact cycles, with the topology induced from $Z_d(X)$. (p. **21**)
$\mathcal{B}_d(X)$	Barlet space (p. **21**)
$Z^{(d)}$	Pure d-dimensional part of a cycle Z (p. **17**)
$B_{\mathcal{S}}(k)$	Scale neighbourhood (p. **15**, **17**)
$B_{\mathcal{S}}$	$\bigcup_{k\geq 0} B_{\mathcal{S}}(k)$ (p. **17**)
$Z = \lim_{k\to\infty} Z_k$	The sequence (Z_k) of cycles converges to the cycle Z in comparison to the Barlet topology (p. **15**)
$A = \lim_{k\to\infty} A_k$	The sequence (A_k) of closed sets converges set theoretically to the closed set A (p. **16**)
$f_*\colon Z_*(X) \longrightarrow Z_*(Y)$	The push-forward of a mapping $f\colon X \longrightarrow Y$ (p. **17**)
$\varphi_t^\sharp\colon B_{\mathcal{S}} \longrightarrow \mathbb{C}$	A special complex-valued function construct from a mapping $\varphi\colon X \longrightarrow \mathbb{C}$ (p. **18**)
$\mathrm{Sym}^k(W)$	The quotient of W^k by the group of permutations S_k (p. **19**)
$\mu_f(S)$	Multiplicity of the irreducible component S of a fibre of f (p. **22**)
$Y_{\mathrm{gen}(f)}$	generic locus of a mapping $f\colon X \longrightarrow Y$ (p. **23**)
$Z_{f,\mathrm{gen}}\colon Y_{\mathrm{gen}(f)} \longrightarrow Z_d(X)$	Mapping associating to each point of $Y_{\mathrm{gen}(f)}$ a cycle, whose support is the fibre on the point (p. **23**)
$Z_f\colon Y \longrightarrow Z_*(X)$	Continuous extension of $Z_{f,\mathrm{gen}}$ in case where f is geometrically flat (p. **26**)
$Z(f)$	Fibre-cycle space of f (p. **24**)
${}_{Z(f)}\mathcal{O}$	Structure sheaf on $Z(f)$ (p. **25**)
Φ_R	Meromorphic quotient of the meromorphic equivalence relation R (p. **30**)
(U, f, V)	Local regular foliation (p. **32**)
$\mathcal{A}_{\mathbb{F}}$	Atlas of a foliation $\mathbb{F}$ (p. **32**)
$\dim \mathbb{F}$ ($\operatorname{codim} \mathbb{F}$)	Dimension (codimension) of a foliation $\mathbb{F}$ (p. **32**)
L_x	Leaf through a point $x \in X$ (p. **33**)

$R^{\mathbb{F}}$	Equivalence relation whose classes are leaves of $\mathbb{F}$ (p. **33**, **35**)
$X/\mathbb{F}$, $X^\rho/\mathbb{F}$	Leaf space of $\mathbb{F}$ (p. **33**, **35**)
$A/\mathbb{F}$	Quotient of a $\mathbb{F}$-saturated subset of X by $R^{\mathbb{F}}$ (p. **33**)
(U, p)	Local $\mathbb{F}$-foliation satisfying definition 3.1.5 (p. **33**)
$\operatorname{Sing}\mathbb{F}$	Singular locus of $\mathbb{F}$ (p. **34**)
$X^{\mathrm{reg}}(\mathbb{F})$, X^{reg}	Regular set of $\mathbb{F}$ (p. **34**)
$\mathbb{F}^{\mathrm{reg}}$	$\mathbb{F}\vert_{X^{\mathrm{reg}}(\mathbb{F})}$ (p. **34**)
$\Theta^{\mathbb{F}}$ $(\Omega^{\mathbb{F}})$	Sheaf of germs of vector fields (1-forms) associated to the foliation (p. **34**)
$X^\rho(\mathbb{F})$ (X^ρ)	Set of points through which a local leaf of $\mathbb{F}$ passes (p. **35**)
$\Sigma(\mathbb{F})$	$X \smallsetminus X^\rho(\mathbb{F})$ (p. **35**)
$\mathrm{Sh}\mathbb{F}$	Singular hull of $\mathbb{F}$ (p. **36**)
$X^{\mathrm{ns}}(\mathbb{F})$, X^{ns}	$X \smallsetminus \overline{\mathrm{Sh}\mathbb{F}}$ (p. **37**)
$\mathbb{F}^{\mathrm{ns}}$	$\mathbb{F}\vert_{X^{\mathrm{ns}}(\mathbb{F})}$ (p. **37**)
$\mathcal{H}_1(T)$	$\{x \in T \mid \exists U \Subset T$ of x such that $\overline{U}$ is Haudorff$\}$ (p. **39**)
$\mathcal{H}_2(T)$	$\{x \in T \mid \forall\ y \neq x$, x and y are separable by open subsets of $T\}$ (p. **39**)
$\nu_R\colon T \longrightarrow \mathbb{N}_{>0}$	$\nu_R(x) := \operatorname{Card} R(x)$ (p. **42**)
$M_R(U)$	$\max_{v\in U}(\nu_{R\vert_U}(v))$ (p. **42**)
$\mu_R(x)$	$\min_{x\in U\Subset T}(M_R(U))$ (p. **42**)
X^R_{tr}	$\{x \in X \mid \mu_R(x) = 1\}$ (p. **44**)
$\mathcal{K}(X)$	Set of compact subsets of X (p. **46**)
$d_{\mathcal{H}}$	Hausdorff metric on $\mathcal{K}(X)$ (p. **46**)
$U_\varepsilon(K)$	$\{x \in X \mid d(x, K) < \varepsilon\}$ (p. **46**)
$B^{\mathcal{H}}_\varepsilon(K)$	$\{L \in \mathcal{K}(X) \mid d_{\mathcal{H}}(K, L) < \varepsilon\}$ (p. **46**)
$\widetilde{Z}_d(X)$	Set of analytic subsets of X of pure dimension d (p. **50**)
R_O	Equivalence relation on $Z_d(X)$ that forgets the multiplicities of the cycles (p. **50**)
$X^1_{\mathrm{st}}(\mathbb{F})$, X^1_{st}	$\{x \in X \mid L_x$ is 1-stable$\}$ (p. **51**)
$X^2_{\mathrm{st}}(\mathbb{F})$, X^1_{st}	$\{x \in X \mid L_x$ is 2-stable$\}$ (p. **51**)

R_p	Equivalence relation on $V := p(U)$ with the property that $p^{-1}(R_p(p(x))) = L_x \cap U$ (where (U,p) is a local $\mathbb{F}$-foliation) (p. **52**)
$\nu_p : p(U) \longrightarrow \mathbb{N}_{>0}$	$\nu_p(v) := \nu_{R_p}(v) = \operatorname{Card} R_p(v)$ (p. **53**)
$\mu_t(L_x)$	Topological multiplicity of L_x (p. **55**)
$\mu_a(L_x)$	Analytical multiplicity of L_x (p. **57**)
$G(\mathbb{F})$, G	Good set of $\mathbb{F}$ (p. **53**, **66**)
$X_{\mathrm{tr}}(\mathbb{F})$, X_{tr}	Trivial locus of $\mathbb{F}$ (p. **56**, **66**)
$C(\mathbb{F})$, C	Interior of $\{x \in G \mid \zeta_{\mathbb{F}}$ is continuous in $x\}$ (p. **59**, **78**)
$X^{\mathrm{cl}}(\mathbb{F})$, X^{cl}	$\{x \in X^{\mathrm{ns}} \mid L_x$ is closed in $X\}$ (p. **83**)
$C^{\mathrm{cl}}(\mathbb{F})$, C^{cl}	Interior of $\{x \in G \cap X^{\mathrm{cl}} \mid \zeta_{\mathbb{F}}^X$ is continuous in $x\}$ (p. **83**)
$\zeta_{\mathbb{F}} : G \longrightarrow Z_d(X^\rho)$	$\zeta_{\mathbb{F}}(x) := \mu_t(L_x)[L_x]$ (p. **59**, **66**)
$\zeta_{\mathbb{F}}^X : G \cap X^{\mathrm{cl}} \longrightarrow Z_d(X)$	$\zeta_{\mathbb{F}}^X(x) := \mu_t(L_x)[L_x]$ (p. **83**)
R_C	$R^{\mathbb{F}^{\mathrm{ns}}}\vert_C$ (p. **78**)
$M^{\mathbb{F}}$	$\overline{R_C}$ (p. **78**)
$Z(\mathbb{F})$	Meromorphic leaf space of $\mathbb{F}$ (p. **81**)

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CURRICULUM VITAE

Education

Since 1998	**Assistant** at the Department of Mathematics of the University of Fribourg
Since 1998	**Doctoral studies** in mathematics (complex analysis) at the University of Fribourg supervised by Prof. Burchard Kaup.
1993 - 1997	**Diploma** in mathematics with computer sciences as minor branch.
1993	*Baccalauréat*, type C, obtained at College St-Michel of Fribourg

Professional experiences

Since September 1999, **manager** of the computers of the Department of Mathematics of the University of Fribourg (35 Macintoshes and 2 PCs).

Since February 1998, **assistant** in Mathematics with tasks: supervise of exercises' session (up to 100 students), stand in for the professor punctually, take part at the organization of international congresses in mathematics (summer 1998 in geometry and summer 2000 in algebra). Since September 2000, head of the assistants of the department.

Since September 1995, **teacher** of mathematics at the school SOFT (technical school of computer sciences).

Trainee in the computer system at the Swiss Federal Research Station for Agronomy at Posieux (July and August 1995, August 1996, January and February 1997).

Supply teacher of mathematics at College St-Michel in Fribourg (September 1995 and 1996).

Supplementary knowledge

Good knowledge in computer sciences: programming in C, C++, Pascal, Matlab, Mathematica or Eiffel, and managing of computers (PC or Mac). Knowledge in Unix.

Non professional activities

Since 1997, accountant of the Wind Band "La Cordiale" of Neyruz.

Since 1984, amateur musician (since 1986 in the Conservatoire of Fribourg, since 1986 at "La Cordiale" of Neyruz, from 1985 to 1991 at the Landwehr of Fribourg).

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